Totally geodesic submanifolds of Nearly Kähler and G_2-manifolds

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Preliminaries

- M real analytic Riemannian manifold.
- Metric $\langle \cdot, \cdot \rangle \colon T_pM \times T_pM \to \mathbb{R}$.
- Levi-Civita connection $\nabla \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$.
- Curvature tensor $R \colon T_pM \times T_pM \times T_pM \to T_pM$:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

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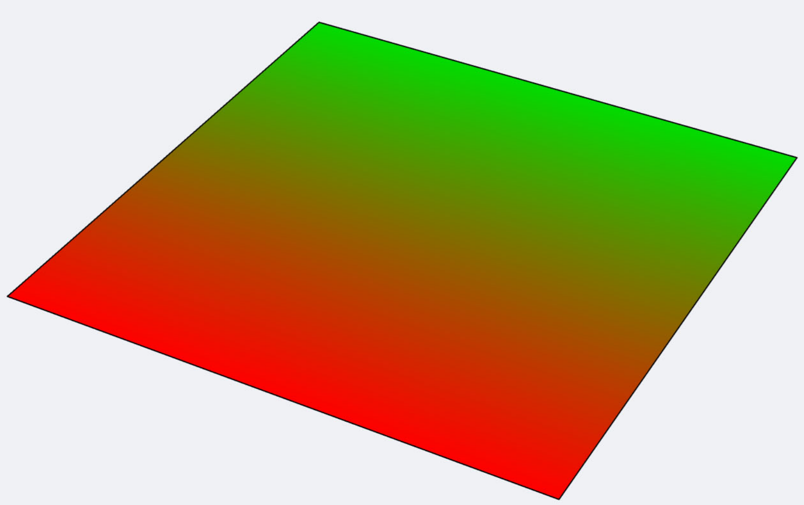
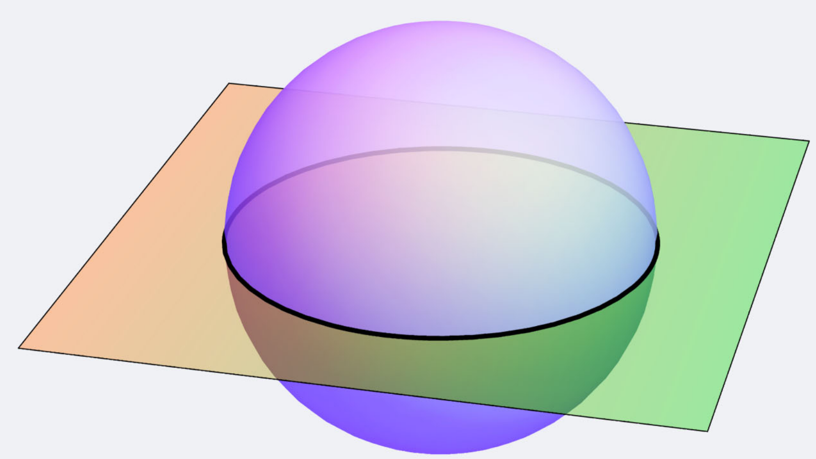
- f is *totally geodesic* if:

$$\gamma\colon I \to \Sigma \text{ geodesic} \Rightarrow f \circ \gamma\colon I \to M \text{ geodesic.}$$



$\mathbb{R}^k \subseteq \mathbb{R}^n$



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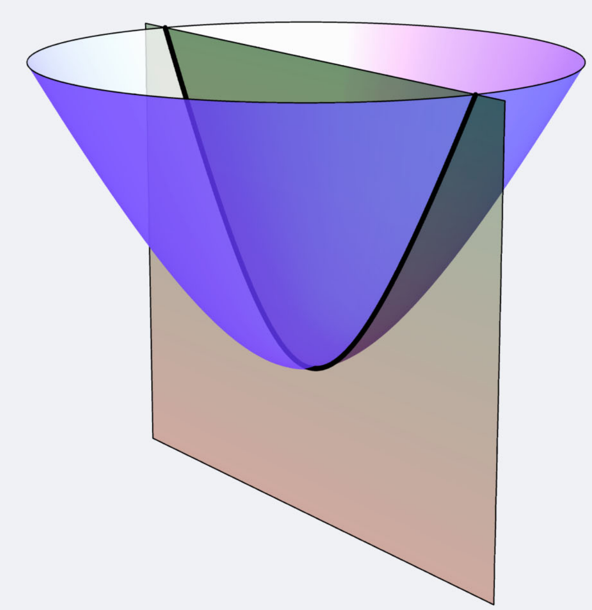

$\mathbb{R}\mathsf{H}^k \subseteq \mathbb{R}\mathsf{H}^n$

General problem

- $f\colon \Sigma^k \to M^n$ is *compatible* if $\widetilde{f}\colon \Sigma \to \mathsf{Gr}_k(TM)$ given by

$$\widetilde{f}(x) = df_x(T_x\Sigma)$$

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- $f_i\colon \Sigma_i \to M$ $(i = 1, 2)$ are *equivalent* (and we write $f_1 \simeq f_2$) if there exists an isometry $\phi\colon \Sigma_1 \to \Sigma_2$ with $f_1 = f_2 \circ \phi$.

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Given M, classify (equivalence classes of) inextendable compatible totally geodesic immersions to M up to congruence.

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- Butruille ('06): Classification of homogeneous nearly Kähler manifolds G/K in dimension six:

M	G	K
S^6	G_2	$\mathsf{SU}(3)$
$\mathsf{F}(\mathbb{C}^3)$	$\mathsf{SU}(3)$	T^2
$\mathbb{C}\mathsf{P}^3$	$\mathsf{Sp}(2)$	$\mathsf{U}(1) \times \mathsf{Sp}(1)$
$\mathsf{S}^3 \times \mathsf{S}^3$	$\mathsf{SU}(2)^3$	$\Delta\mathsf{SU}(2)$

Previously known results

- $\mathbb{C}\mathsf{P}^3$:
  - Totally geodesic + Lagrangian (Aslan '23, Liefsoens '22).
  - Totally geodesic + J-holomorphic curve (Cwiklinski, Vrancken '22).
- $\mathsf{F}(\mathbb{C}^3)$:
  - Totally geodesic + Lagrangian (Storm '20).
- $\mathsf{S}^3 \times \mathsf{S}^3$:
  - Totally geodesic + Lagrangian (Zhang, Dioos, Hu, Vrancken, Wang '16).
  - Totally geodesic + J-holomorphic curve (Bolton, Dillen, Dioos, Vrancken '22).

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M is ***reductive*** if there exists $\mathfrak{p} \subseteq \mathfrak{g}$ with

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- Isotropy representation $\mathsf{K} \curvearrowright T_o M \leftrightsquigarrow$ Adjoint action $\mathsf{K} \curvearrowright \mathfrak{p}$.
- G-invariant tensors on $M \leftrightsquigarrow \mathrm{Ad}(\mathsf{K})$-invariant tensors on $\mathfrak{p}$.

Connection and curvature

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$$\begin{aligned} D_X Y &= \frac{1}{2}[X, Y]_{\mathfrak{p}} + U(X, Y), \\ 2\langle U(X, Y), Z\rangle &= \langle [Z, X]_{\mathfrak{p}}, Y\rangle + \langle X, [Z, Y]_{\mathfrak{p}}\rangle, \\ (\nabla_{X^*} Y^*)_o &= -[X, Y]_{\mathfrak{p}} + D_X Y, \\ R(X, Y) &= [D_X, D_Y] - D_{[X,Y]_{\mathfrak{p}}} - \mathrm{ad}([X, Y]_{\mathfrak{k}}). \end{aligned}$$

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- M is naturally reductive $\Leftrightarrow \mathrm{Exp}(tX) \cdot o$ geodesic for all $X \in \mathfrak{p}$.
- M is *normal homogeneous* if:

1. G has a bi-invariant metric.
2. The complement $\mathfrak{p} = \mathfrak{g} \ominus \mathfrak{k}$.
3. The metric on M is induced from the metric on G.

Totally geodesic subspaces

- $f\colon \Sigma \to M$ totally geodesic. Then f is determined by any tangent subspace $V \in \widetilde{f}(\Sigma)$:

$$f_i\colon \Sigma_i \to M, \quad \widetilde{f_1}(\Sigma_1) \cap \widetilde{f_2}(\Sigma_2) \neq \varnothing \Rightarrow f_1 \simeq f_2.$$

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Theorem (Cartan '51, Hermann '59). M real analytic, $V \subseteq T_pM$. The following are equivalent:

1. V is a totally geodesic subspace.
2. V is $\nabla^k R$-invariant for all $k \geq 0$.

D-invariant totally geodesic submanifolds

- $M = \mathsf{G}/\mathsf{K}$ naturally reductive, $\Sigma \to M$ submanifold.
- Σ is D-*invariant* if $T_x\Sigma$ is D-invariant for all $x \in \Sigma$.

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Theorem (LN, Rodríguez-Vázquez). $\mathfrak{v} \subseteq \mathfrak{p}$ subspace. The following are equivalent:

1. $\mathfrak{v}$ is invariant under R and D.
2. $\mathfrak{v}$ is invariant under R^c and D.
3. There exists a D-invariant t.g. $\Sigma \to M$ with $T_o\Sigma = \mathfrak{v}$.

The corresponding t.g. submanifold is $\Sigma = \mathsf{S} \cdot o$, where $\mathfrak{s} = [\mathfrak{v}, \mathfrak{v}] + \mathfrak{v}$.

$$\mathbb{C}\mathsf{P}^3 = \mathsf{Sp}(2)/(\mathsf{U}(1) \times \mathsf{Sp}(1))$$

Submanifold	Orbit of	Relationship with J
$\mathbb{R}\mathsf{P}^3_{\mathbb{C},1/2}(\sqrt{2})$	$\mathsf{SU}(2)$	Lagrangian
$\mathsf{S}^2(1/\sqrt{2})$	$\mathsf{Sp}(1)_f$	J-holomorphic
$\mathsf{S}^2(1)$	$\mathsf{SU}(2)$	J-holomorphic
$\mathsf{S}^2(\sqrt{5})$	$\mathsf{SU}(2)_{\Lambda_3}$	J-holomorphic

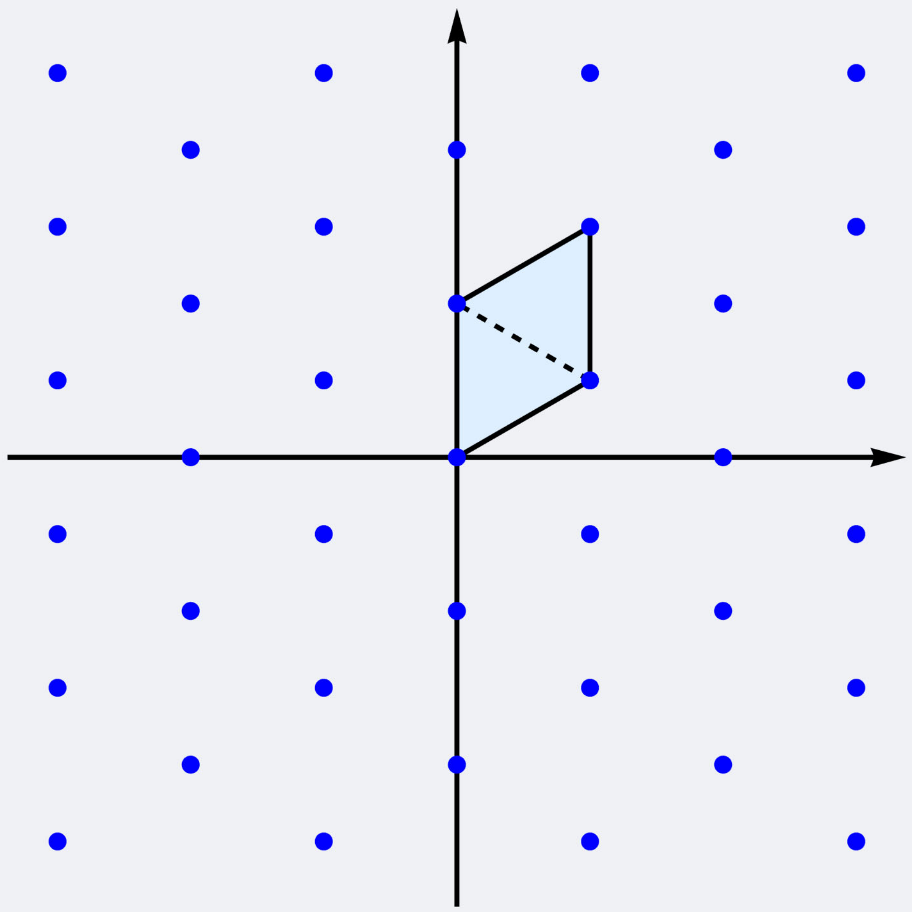
$\Lambda_3 = \mathsf{S}^3(\mathbb{C}^2)$ is the four-dimensional irrep. of $\mathsf{SU}(2)$.

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \mapsto \begin{pmatrix} a(|a|^2 - 2|b|^2) & -\sqrt{3}a^2\bar{b} \\ \sqrt{3}a^2\bar{b} & a^3 \end{pmatrix} + \mathbf{j} \begin{pmatrix} b(2|a|^2 - |b|^2) & -\sqrt{3}ab^2 \\ \sqrt{3}\bar{a}b^2 & -b^3 \end{pmatrix}$$

$\mathsf{F}(\mathbb{C}^3) = \mathsf{SU}(3)/\mathsf{T}^2$

Submanifold	Orbit of	Relationship with J
$\mathsf{F}(\mathbb{R}^3) = \mathsf{S}^3(2\sqrt{2})/\mathsf{Q}_8$	$\mathsf{SO}(3)$	Lagrangian
$\mathsf{S}^3_{\mathbb{C},1/4}(2)$	$\mathsf{SU}(2)$	Lagrangian
T_Λ	T^2	J-holomorphic
$\mathsf{S}^2(1/\sqrt{2})$	$\mathsf{U}(2)$	J-holomorphic
$\mathsf{S}^2(\sqrt{2})$	$\mathsf{SO}(3)$	J-holomorphic
$\mathbb{R}\mathsf{P}^2(2\sqrt{2})$	Inhomogeneous	Totally real

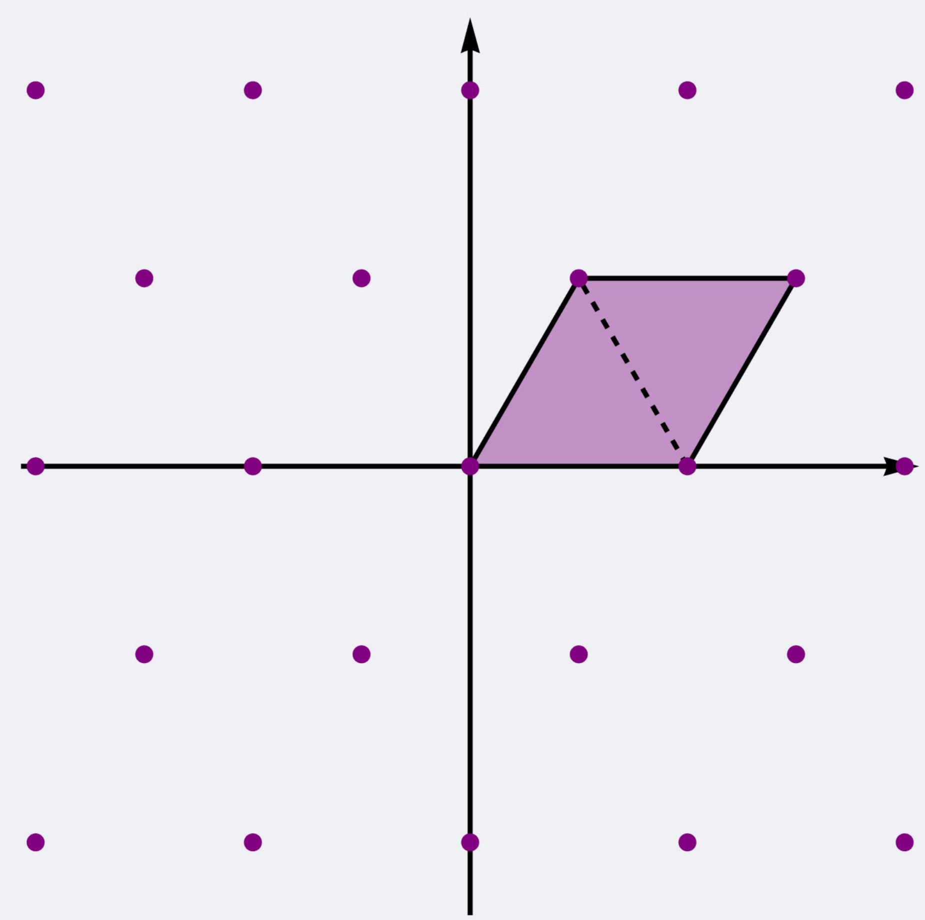
$$\Lambda = \left\langle \sqrt{\tfrac{2}{3}}\pi\,(0,2), \sqrt{2}\pi\left(1, \tfrac{1}{\sqrt{3}}\right) \right\rangle$$



$\mathsf{S}^3 \times \mathsf{S}^3 = \mathsf{SU}(2)^3/\Delta\mathsf{SU}(2)$

Submanifold	Orbit of	Relationship with J
$\mathsf{S}^3(2/\sqrt{3})$	$\mathsf{SU}(2)_2$	Lagrangian
$\mathsf{S}^3_{\mathbb{C},1/3}(2)$	$\mathsf{SU}(2)_{13,2}$	Lagrangian
T_Γ	$\mathsf{T} \subseteq \mathsf{U}(1)^3$	J-holomorphic
$\mathsf{S}^2(\sqrt{3/2})$	$\Delta\mathsf{SU}(2)$	J-holomorphic
$\mathsf{S}^2(2/\sqrt{3})$	$\mathsf{H} \subseteq \mathsf{SU}(2)_{13,2}$	Totally real

$$\Gamma = \left\langle \tfrac{4\pi}{\sqrt{3}}(1,0),\, \tfrac{2\pi}{\sqrt{3}}(1,\sqrt{3}) \right\rangle$$



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- If $\dim M = 6$, and $\mathrm{Ric} = 5g$, then (Bär '93):

$$M \text{ is strict nearly Kähler} \Rightarrow \mathrm{Hol}(\widehat{M}) \in \{0, \mathsf{G}_2\}.$$

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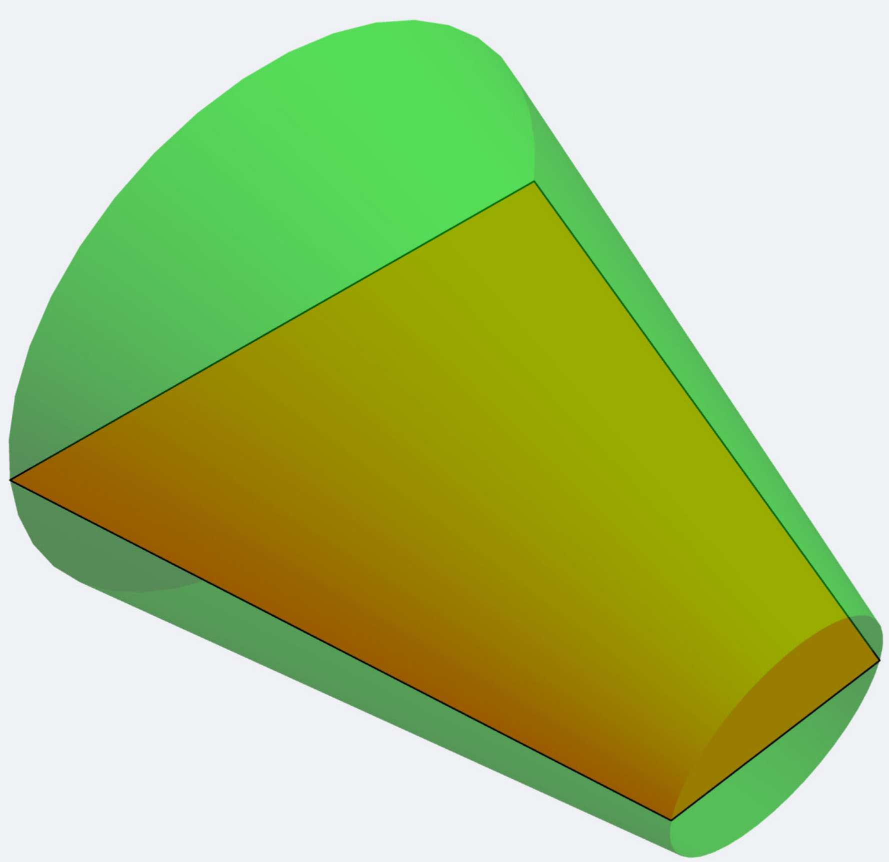
- If $M \neq \mathsf{S}^6$ and $\pi_1(M) = 0$, then $\mathrm{Hol}(\widehat{M}) = \mathsf{G}_2$.

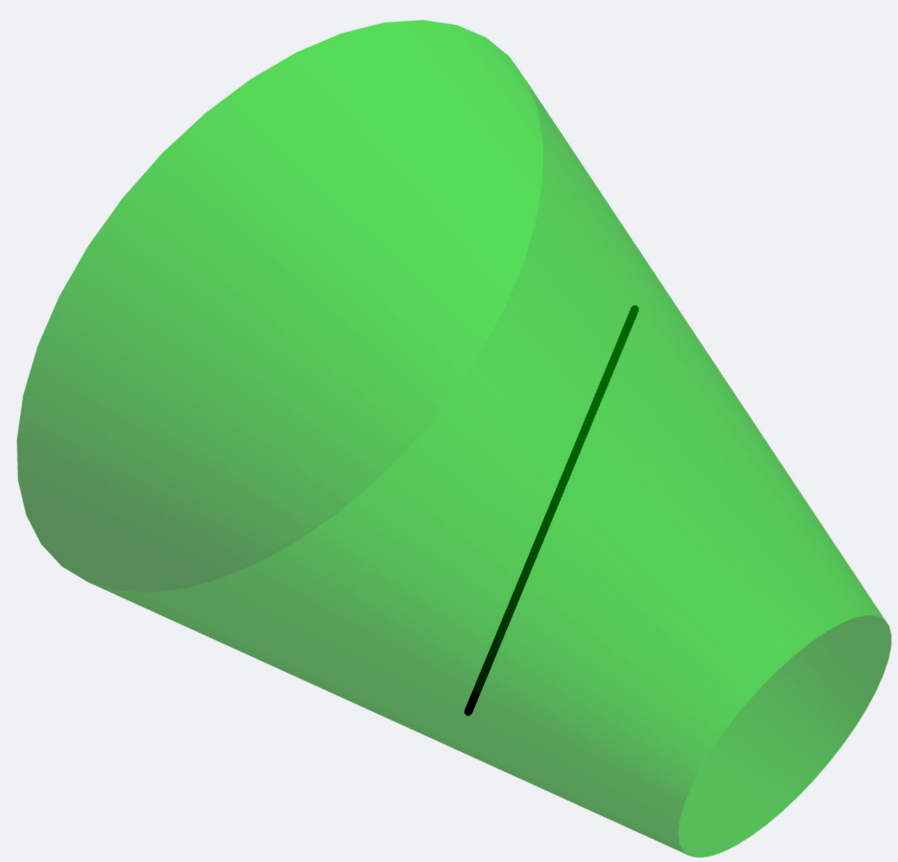
Theorem (LN, Rodríguez-Vázquez). Σ totally geodesic in $\widehat{M}$. Then one of the two holds:

i. $\Sigma = \widehat{S}$ for $S \to M$ totally geodesic.

ii. Σ is (up to surjective local isometry) a totally geodesic hypersurface in $\widehat{S}$ for $S \to M$ totally geodesic.



$\Sigma = \widehat{S}$



$\Sigma \subseteq \widehat{S}$

Corollary. $\Sigma \to \widehat{M}$ maximal totally geodesic submanifold. Then one of the two holds:

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Example of a type ii submanifold. Take $M = (0, \pi/2) \times \mathbb{R}^2$ with the metric

$$dx^2 + (\sin x)^2 p(y,z)^2 dy^2 + (\sin x)^2 q(y,z)^2 dz^2$$

The hypersurface

$$\Sigma = \Big\{ (\sec x, x, y, z) \colon (x,y,z) \in M \Big\} \subseteq \widehat{M}$$

is totally geodesic.

The case ii. is not possible if $\mathrm{Hol}(\widehat{M}) = \mathsf{G}_2$ (Jentsch, Moroianu, Semmelmann, '13).

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i. Associative (i.e. calibrated by the G_2-structure ϕ) if $\dim \Sigma = 3$.

ii. Coassociative (i.e. calibrated by $\star\phi$) if $\dim \Sigma = 4$.

Ambient	Submanifold	Orbit of	Relationship with J
$\mathbb{C}\mathsf{P}^3$	$\mathbb{R}\mathsf{P}^3_{\mathbb{C},1/2}(\sqrt{2})$	$\mathsf{SU}(2)^j$	Lagrangian
$\mathbb{C}\mathsf{P}^3$	$\mathsf{S}^2(1/\sqrt{2})$	$\mathsf{Sp}(1)_f$	J-holomorphic
$\mathbb{C}\mathsf{P}^3$	$\mathsf{S}^2(1)$	$\mathsf{SU}(2)$	J-holomorphic
$\mathbb{C}\mathsf{P}^3$	$\mathsf{S}^2(\sqrt{5})$	$\mathsf{SU}(2)_{\Lambda_3}$	J-holomorphic
$\mathsf{F}(\mathbb{C}^3)$	$\mathsf{F}(\mathbb{R}^3)$	$\mathsf{SO}(3)$	Lagrangian
$\mathsf{F}(\mathbb{C}^3)$	$\mathsf{S}^3_{\mathbb{C},1/4}(2)$	$\mathsf{SU}(2)$	Lagrangian
$\mathsf{F}(\mathbb{C}^3)$	T_Λ	T^2	J-holomorphic
$\mathsf{F}(\mathbb{C}^3)$	$\mathsf{S}^2(1/\sqrt{2})$	$\mathsf{U}(2)$	J-holomorphic
$\mathsf{F}(\mathbb{C}^3)$	$\mathsf{S}^2(\sqrt{2})$	$\mathsf{SO}(3)$	J-holomorphic
$\mathsf{F}(\mathbb{C}^3)$	$\mathbb{R}\mathsf{P}^2(2\sqrt{2})$	Inhomogeneous	Totally real
$\mathsf{S}^3 \times \mathsf{S}^3$	$\mathsf{S}^3(2/\sqrt{3})$	$\mathsf{SU}(2)_2$	Lagrangian
$\mathsf{S}^3 \times \mathsf{S}^3$	$\mathsf{S}^3_{\mathbb{C},1/3}(2)$	$\mathsf{SU}(2)_{13,2}$	Lagrangian
$\mathsf{S}^3 \times \mathsf{S}^3$	T_Γ	$\mathsf{T} \subseteq \mathsf{U}(1)^3$	J-holomorphic
$\mathsf{S}^3 \times \mathsf{S}^3$	$\mathsf{S}^2(\sqrt{3/2})$	$\Delta\mathsf{SU}(2)$	J-holomorphic
$\mathsf{S}^3 \times \mathsf{S}^3$	$\mathsf{S}^2(2/\sqrt{3})$	$\mathsf{H} \subseteq \mathsf{SU}(2)_{13,2}$	Totally real