Polar homogeneous foliations on hyperbolic spaces

Juan Manuel Lorenzo Naveiro
University of Oklahoma
Symmetry and Shape, 05/11/2025

Joint work with J. C. Díaz-Ramos
(Universidade de Santiago de Compostela)

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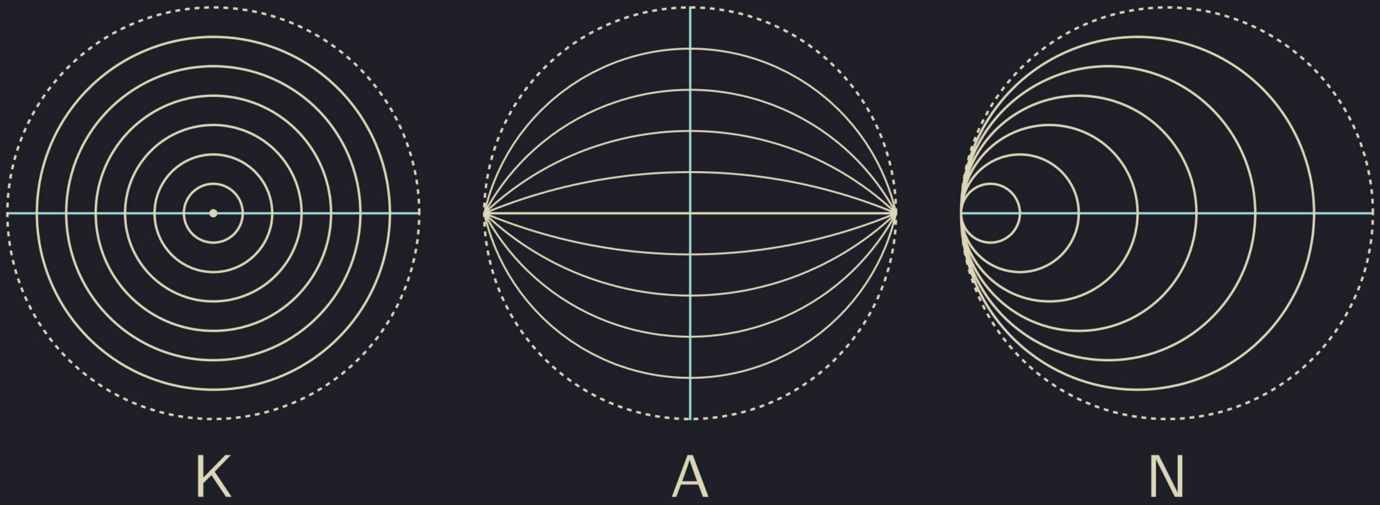
$\mathsf{H} \curvearrowright M$ is *polar* if $\exists\, \Sigma \subseteq M$ with:

- $\Sigma \cap (\mathsf{H} \cdot p) \neq \varnothing$ for all $p \in M$,
- $T_p\Sigma \perp T_p(\mathsf{H} \cdot p)$ for all $p \in \Sigma$.

Σ is a *section*.

Examples on $\mathbb{R}\mathsf{H}^2 = \mathsf{SL}(2, \mathbb{R})/\mathsf{SO}(2)$



Two actions $\mathsf{H}_1 \curvearrowright M,\ \mathsf{H}_2 \curvearrowright M$ are *orbit equivalent* if $\exists\, f \in I(M)$ with $f(\mathsf{H}_1 \cdot p) = \mathsf{H}_2 \cdot f(p)$ for all $p \in M$.

Main problem. Given M, classify polar actions on M up to orbit equivalence.

Known results

- Polar representations: Dadok ('85).

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- $\mathbb{R}\mathrm{H}^n$: Wu ('92).
- Compact rank one symmetric spaces: Podestà, Thorbergsson ('99).
- Compact type, higher rank: Kollross, Lytchak.

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- Codimension two polar homogeneous foliations: Díaz-Ramos, LN ('25).

Symmetric spaces of noncompact type

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Iwasawa decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \qquad \mathsf{G} = \mathsf{KAN}.$$

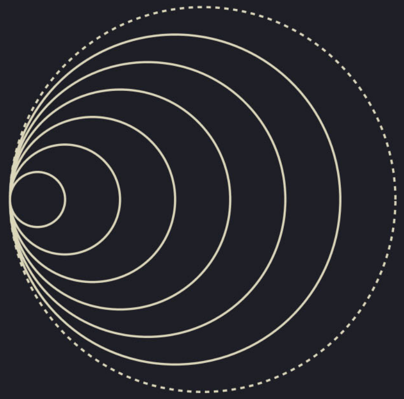
Noncompact rank one symmetric spaces

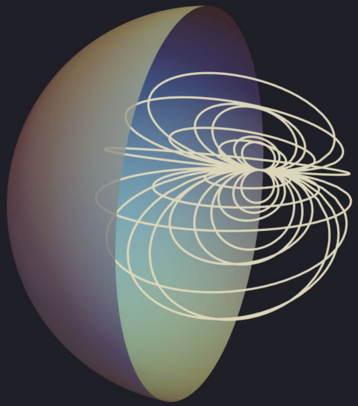
- $M = \mathsf{G}/\mathsf{K} = \mathbb{F}\mathsf{H}^n$ symmetric space of noncompact type and rank one.
- $\mathfrak{a} = \mathbb{R}$ and $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. Moreover, $-1 \leq \sec \leq -\frac{1}{4}$.

M	G	K	$\mathfrak{g}_\alpha$	$\mathfrak{g}_{2\alpha}$
$\mathbb{R}\mathsf{H}^n$	$\mathsf{SO}^0(1,n)$	$\mathsf{SO}(n)$	$\mathbb{R}^{n-1}$	0
$\mathbb{C}\mathsf{H}^n$	$\mathsf{SU}(1,n)$	$\mathsf{S}(\mathsf{U}(1) \times \mathsf{U}(n))$	$\mathbb{C}^{n-1}$	$\mathbb{R}$
$\mathbb{H}\mathsf{H}^n$	$\mathsf{Sp}(1,n)$	$\mathsf{Sp}(1) \times \mathsf{Sp}(n-1)$	$\mathbb{H}^{n-1}$	$\mathbb{R}^3$
$\mathbb{O}\mathsf{H}^2$	F_4^{-20}	$\mathsf{Spin}(9)$	$\mathbb{O}$	$\mathbb{R}^7$

Nilpotent foliation $\mathcal{F}_{\mathfrak{a},\mathfrak{v}}$

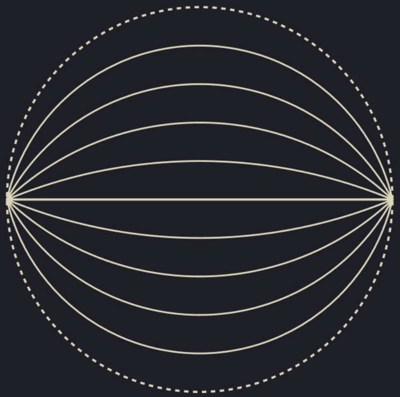
$$\mathfrak{s}_{\mathfrak{a},\mathfrak{v}} = \mathfrak{n} \ominus \mathfrak{v}, \qquad \mathfrak{v} \subseteq \mathfrak{g}_\alpha \text{ abelian subspace.}$$

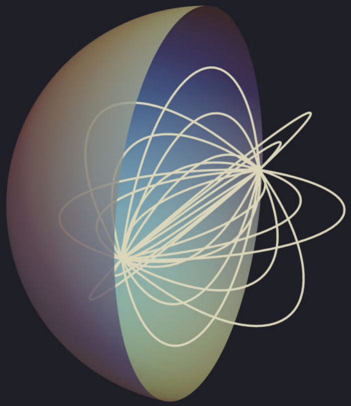




Solvable foliation $\mathcal{F}_{0,\mathfrak{v}}$

$$\mathfrak{s}_{0,\mathfrak{v}} = \mathfrak{a} \oplus (\mathfrak{n} \ominus \mathfrak{v}), \qquad \mathfrak{v} \subseteq \mathfrak{g}_\alpha \text{ abelian subspace.}$$





Theorem (Díaz-Ramos, LN). The hyperbolic planes $M = \mathbb{H}\mathsf{H}^2$ and $\mathbb{O}\mathsf{H}^2$ admit exactly four (nontrivial) polar homogeneous foliations up to orbit equivalence:

$$\mathsf{AN} \curvearrowright M, \quad \mathsf{N} = \mathsf{S}_{\mathfrak{a},0} \curvearrowright M, \quad \mathsf{S}_{0,\ell} \curvearrowright M, \quad \mathsf{S}_{\mathfrak{a},\ell} \curvearrowright M.$$

Remark. $\dim \ell = 1$.

Sections of polar actions $\mathsf{S} \curvearrowright \mathbb{H}\mathsf{H}^n$

- $\mathsf{S} \curvearrowright \mathbb{O}\mathsf{H}^2$ polar action with singular orbits and cohomogeneity ≥ 2
  $\rightsquigarrow$ The section is $\mathbb{R}\mathsf{H}^k(c)$, where $c \in \{-\frac{1}{4}, -1\}$ (Kollross '20).

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Theorem (Díaz-Ramos, LN). If $\mathsf{S} \curvearrowright \mathbb{H}\mathsf{H}^n$ is a nontrivial polar action with cohomogeneity ≥ 2, then its section is a totally geodesic $\mathbb{R}\mathsf{H}^k$ with constant curvature $c \in \{-1, -\frac{1}{4}\}$.

Standard polar foliations on $\mathbb{H}\mathsf{H}^n$

$\mathscr{F}$ homogeneous foliation on M.
$\mathscr{F}$ is *standard* if it is induced by some $\mathsf{S} \subseteq \mathsf{AN}$ up to orbit equivalence.

Note. $\mathsf{S}_{\mathfrak{b},\mathfrak{v}} \curvearrowright \mathbb{F}\mathsf{H}^n$ is standard, where $\mathfrak{s}_{\mathfrak{b},\mathfrak{v}} = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (\mathfrak{n} \ominus \mathfrak{v})$.

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Theorem (Díaz-Ramos, LN). If $\mathscr{F}$ is a (nontrivial) polar standard homogeneous foliation on $\mathbb{H}\mathsf{H}^n$, then $\mathscr{F}$ is isometrically congruent to the orbit foliation induced by some $\mathsf{S}_{\mathfrak{b},\mathfrak{v}}$.

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