Polar actions on some noncompact symmetric spaces

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Workshop on Manifolds with Symmetries, 2022

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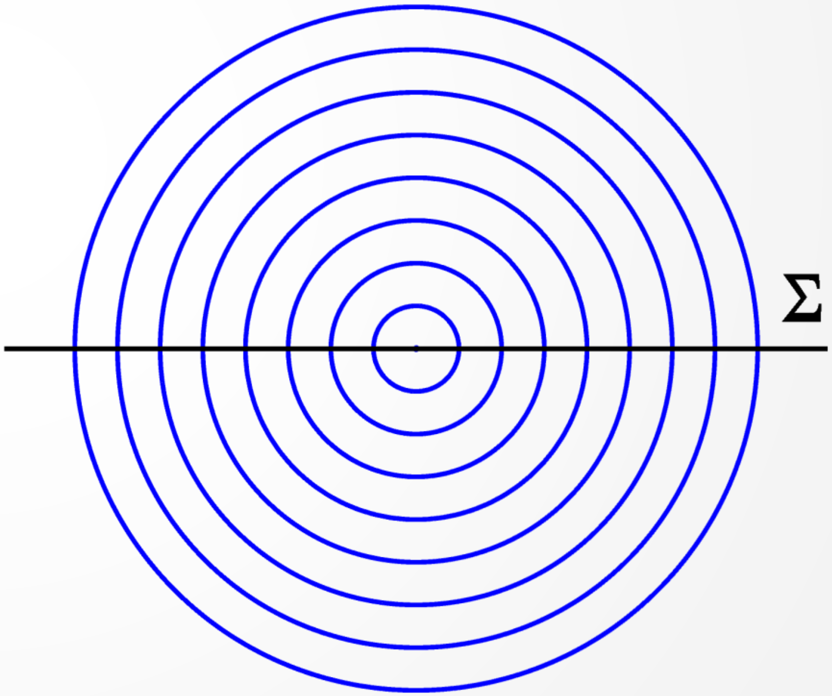

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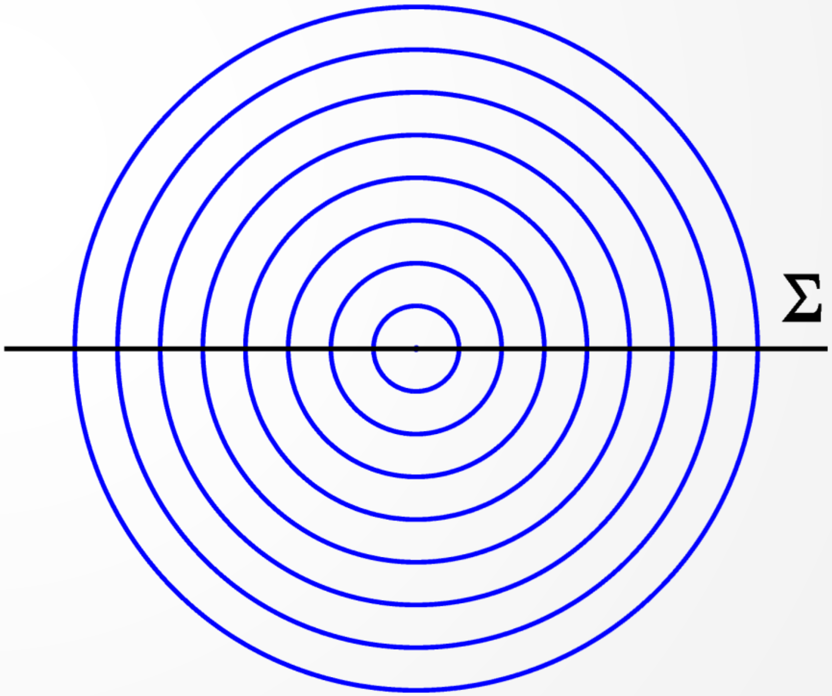
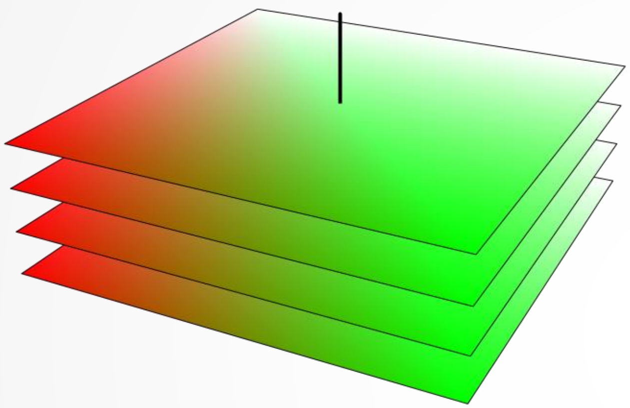
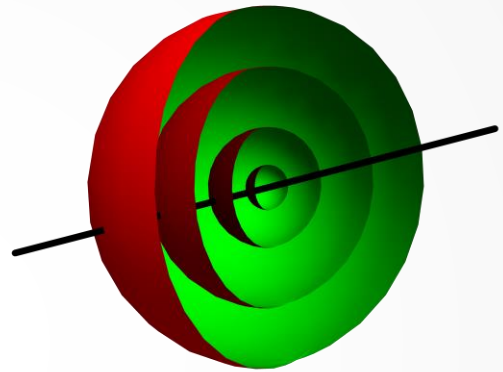
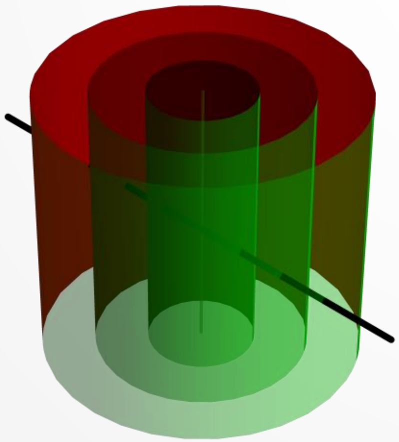


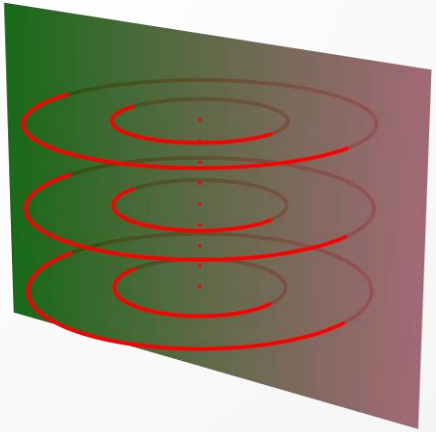
- $\dim(\Sigma) = \mathrm{cohom}(G \curvearrowright M)$.
- Σ is totally geodesic.

Examples on $\mathbb{R}^3$

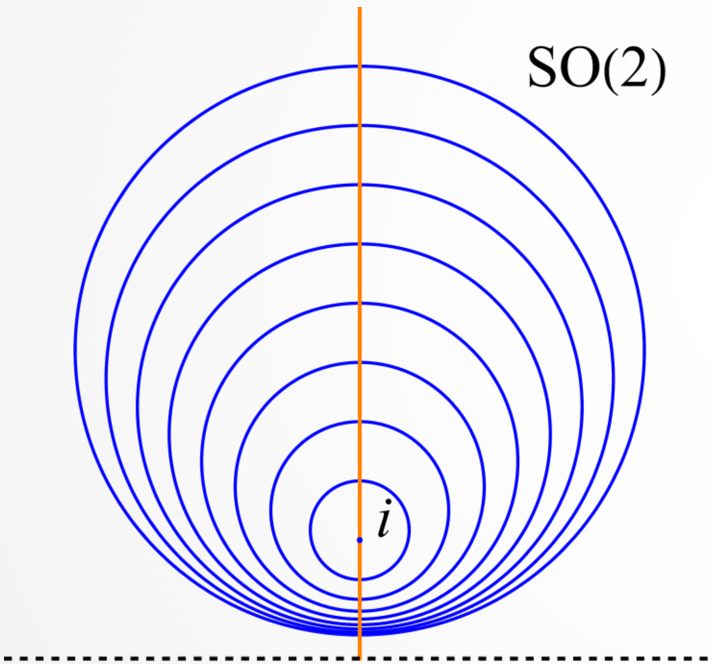


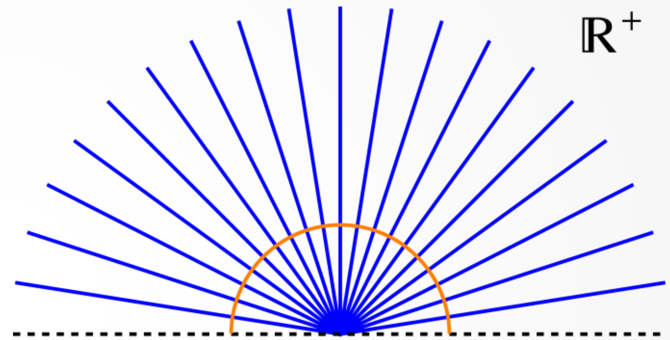


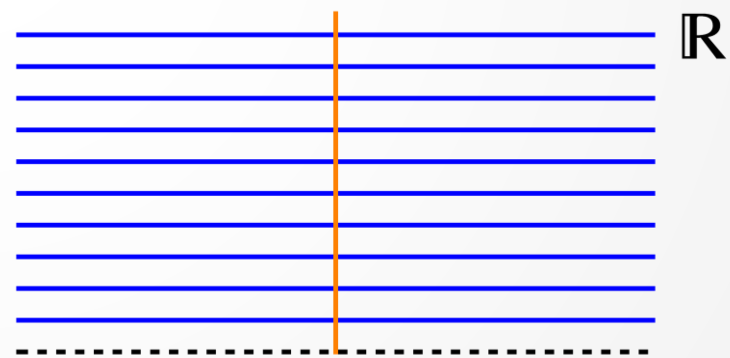




Examples on $\mathbb{R}H^2$







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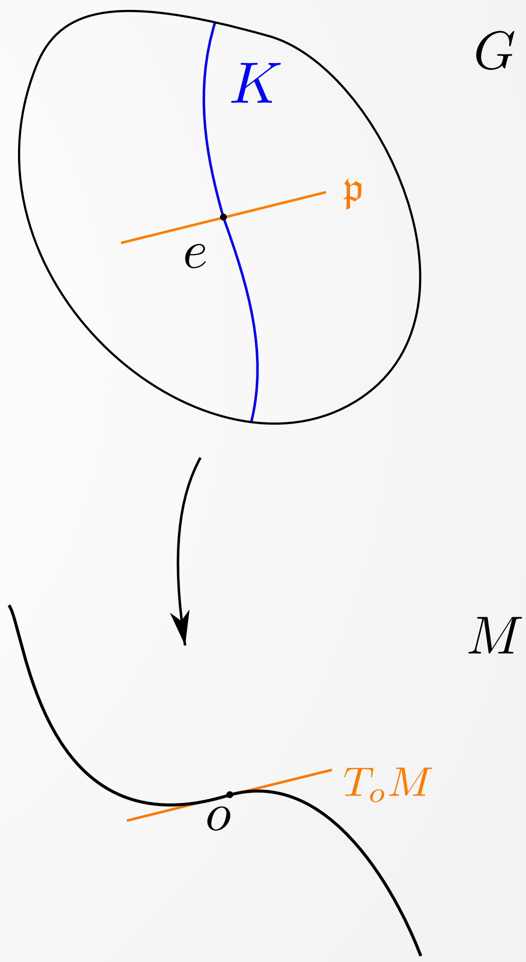

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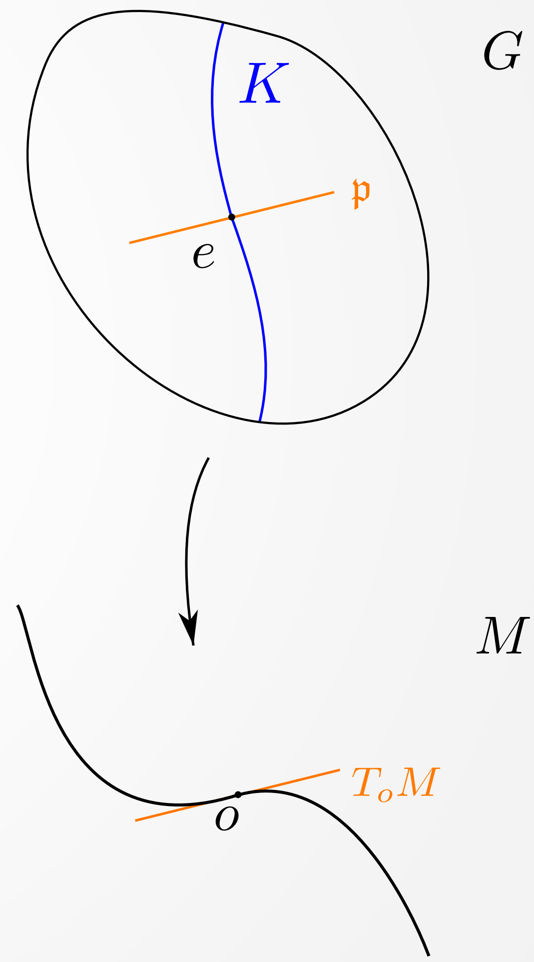
- $\langle X, Y \rangle = -B(X, \theta Y)$ inner product on $\mathfrak{g}$.
- Totally geodesic submanifolds $\leftrightsquigarrow$ Lie triple systems $V \subseteq \mathfrak{p} : \big[[V, V], V\big] \subseteq V$.



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- The slice representation is polar with section $T_o\Sigma$. Can we go backwards?

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- $V = \nu_\xi(H_o \cdot \xi), \quad \Sigma = \exp_o(V)$.

Theorem

$H \curvearrowright M$ is polar $\Leftrightarrow \big[[V,V],V\big] \subseteq V$ and $[V,V] \perp \mathfrak{h}$.

The action is hyperpolar $\Leftrightarrow [V,V] = 0$.

The space $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$

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- $\theta X = -X^T$.
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- $\langle X, Y\rangle$ is proportional to the trace form $\mathrm{tr}(XY^T)$.
- $I^0(M) = \mathrm{SL}(3,\mathbb{R})$.

Iwasawa decomposition

- $\mathfrak{a} = \{\text{Traceless diagonal matrices}\}$.
- $H_{\alpha_1} = \frac{1}{6}\operatorname{diag}(1, -1, 0), \qquad H_{\alpha_2} = \frac{1}{6}\operatorname{diag}(0, 1, -1)$.

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- $\mathfrak{n} = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} = \{\text{Upper triangular matrices}\}$.
- A, N, AN connected subgroups of $\text{SL}(3,\mathbb{R})$ with Lie algebras $\mathfrak{a}, \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}$.

Foliations of cohomogeneity one

- Irreducible case: Berndt—Tamaru.
- Reducible case: Berndt—Díaz-Ramos—Tamaru; Solonenko.

Type $\mathfrak{F}_\ell$
$\mathfrak{h} = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$ $\ell \subseteq \mathfrak{a}$

Type $\mathfrak{F}_1$
$\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}$

Cohomogeneity four

Theorem

An irreducible Riemannian symmetric space contains a totally geodesic hypersurface if and only if it has constant curvature.

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Corollary

$M = \mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ does not admit cohomogeneity four polar actions.

Actions without singular orbits

The general strategy consists of three steps.

1) If $H \leq G$ induces a homogeneous foliation, then $\mathfrak{h}$ is solvable (Berndt—Díaz-Ramos—Tamaru).

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Theorem

If $H \leq G$ acts polarly inducing a cohomogeneity two foliation on $M = G/K$, then $\mathfrak{h}$ is contained in $\mathfrak{b} = \mathfrak{t} \bigoplus \mathfrak{a} \bigoplus \mathfrak{n}$, with $\mathfrak{t} \subseteq Z_{\mathfrak{k}}(\mathfrak{a})$.

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3) Every subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$ induces a homogeneous foliation $\Rightarrow$ apply polarity criterion with $V = \nu_o(H \cdot o)$.

Foliations of cohomogeneity two

- Three foliations up to orbit equivalence

Hyperpolar
$\mathfrak{h} = \mathfrak{n}$
$\mathfrak{h} = \mathbb{R}H_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}$

Non hyperpolar
$\mathfrak{h} = \left(\mathfrak{a} \ominus \mathbb{R}H_{\alpha_1}\right) \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}$

Foliations of cohomogeneity three

/

Unique (non hyperpolar) example
$\mathfrak{h} = \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}$

Theorem: a polar homogeneous foliation $\mathfrak{F}$ on $M = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ is orbit equivalent to the one induced exactly by one of the following subalgebras of $\mathfrak{sl}(3,\mathbb{R})$:

- **Cohomogeneity one**
  - $(\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$
  - $\mathfrak{a} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}$
- **Cohomogeneity two**
  - $\mathfrak{n}$
  - $\mathbb{R}H_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}$
  - $\left(\mathfrak{a} \ominus \mathbb{R}H_{\alpha_1}\right) \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}$
- **Cohomogeneity three**
  - $\mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}$

Actions with singular orbits

- Totally geodesic submanifolds of M are classified (Klein).
- If $H \leq G$ acts polarly with section Σ, we know $[T_o\Sigma, T_o\Sigma] \perp \mathfrak{h}$.

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Proposition

$M = \mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ does not admit cohomogeneity three actions with singular orbits.

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- Totally geodesic submanifolds of M are classified (Klein).
- If $H \leq G$ acts polarly with section Σ, we know $[T_o\Sigma, T_o\Sigma] \perp \mathfrak{h}$.

Proposition

The only polar action with a fixed point on $M = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ is the isotropy action $K \curvearrowright M$.

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