The Cartan and Iwasawa decompositions

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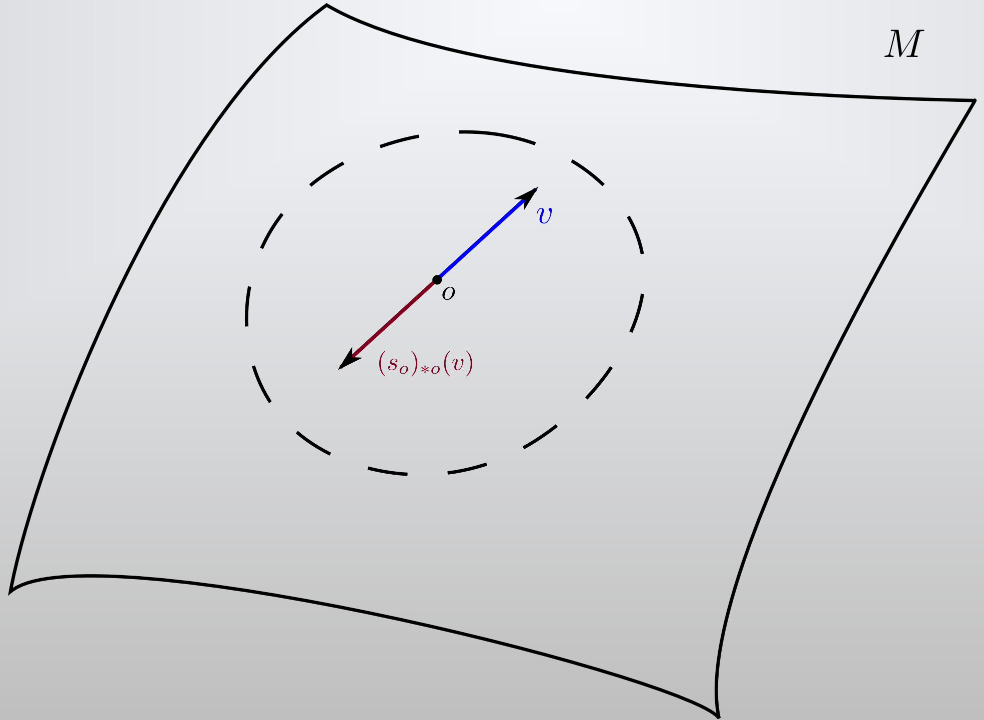
Seminar on Symmetric Spaces, 2022

Symmetric spaces

- M connected Riemannian manifold.
- M is a **symmetric space** if for every $o \in M$ there exists $s_o \in I(M)$.

$$s_o(o) = o, \quad (s_o)_{*o} = -\mathrm{Id}_{T_o M}.$$



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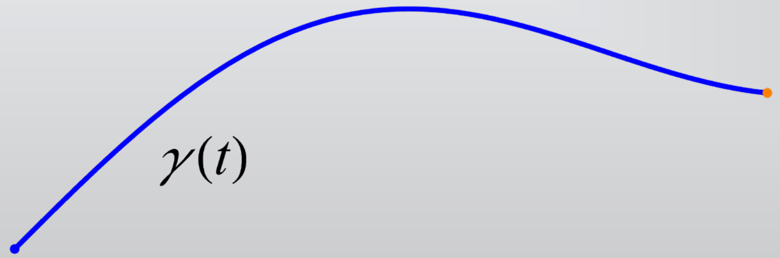
Symmetric spaces are complete.



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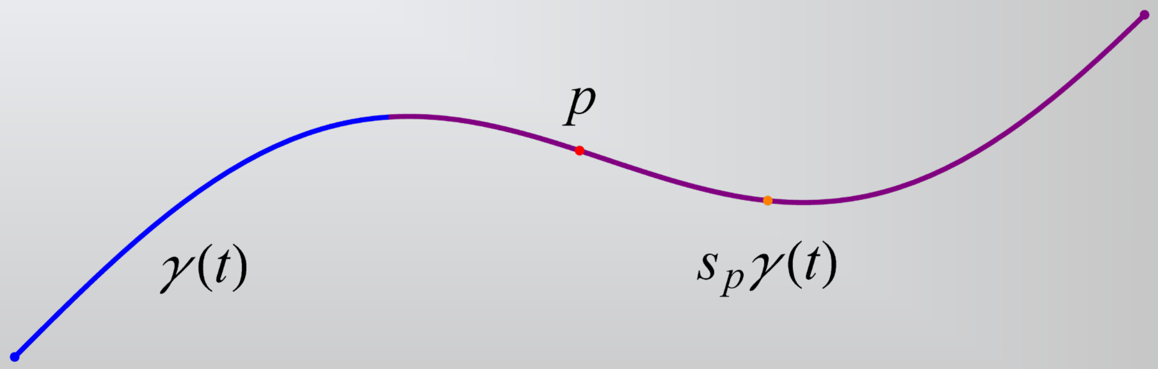

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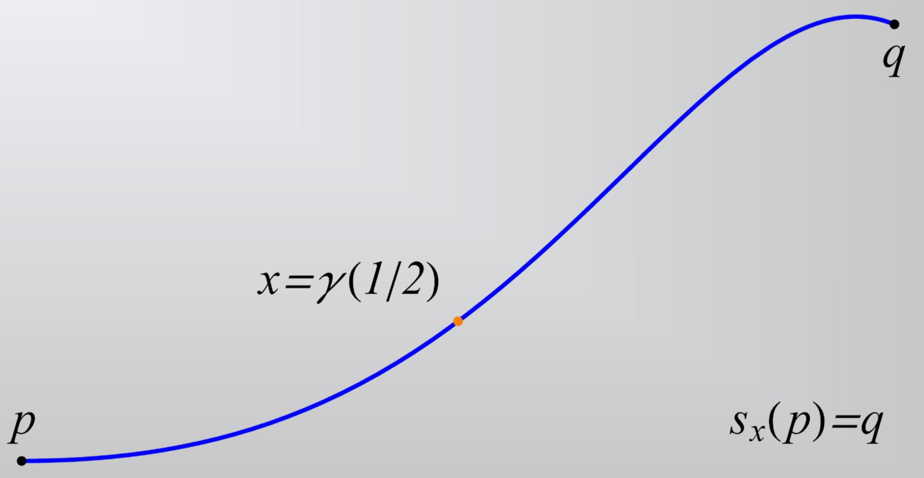

From geometry to algebra

M symmetric space, $o \in M$ fixed.

- $G = I^0(M), K = G_o$. Then $M = G \cdot o = G/K$.
- $\sigma: G \to G$ given by $\sigma(g) = s_o g s_o$.
- $M = G/K$ with $\mathrm{Fix}(\sigma)^0 \subseteq K \subseteq \mathrm{Fix}(\sigma)$.

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Definition

A **symmetric pair** is a triple (G, K, σ) such that:

- G connected Lie group, $K \leq G$ compact subgroup.
- $\sigma: G \to G$ involution, $\mathrm{Fix}(\sigma)^0 \subseteq K \subseteq \mathrm{Fix}(\sigma)$.
- $G \curvearrowright G/K$ is almost effective.

From algebra to geometry

(G, K, σ) symmetric pair, $M = G/K$, $o = eK$.

- $\theta = \sigma_*: \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra involution.

$$\mathfrak{g} = \ker(\theta - 1) \oplus \ker(\theta + 1)\,.$$

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$$\mathfrak{g} = \mathfrak{k} \bigoplus \mathfrak{p}.$$

- $\mathfrak{p} = T_o M$.

$$X \in \mathfrak{p} \mapsto X_o^*, \quad X_p^* = \frac{d}{dt}\Big|_{t=0} \mathrm{Exp}(tX) \cdot p.$$

From algebra to geometry

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$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

- $\mathfrak{p} = T_o M$.
- M is a symmetric space with any G-invariant metric:

$$s_o(gK) = \sigma(g)K.$$

The Cartan decomposition

(G, K, σ) symmetric pair, $M = G/K$, $o = eK$.

Definition

The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the **Cartan decomposition**.
The map $\theta: \mathfrak{g} \to \mathfrak{g}$ is the **Cartan involution**.

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}.$$

Connection and curvature

Proposition

The geodesics of M through o are
$$\exp_o(tX) = \mathrm{Exp}(tX) \cdot o.$$

Proof

Declare $\mathfrak{k} \perp \mathfrak{p} \Rightarrow \pi: g \in G \to g \cdot o \in M$ is a Riemannian submersion.
Let $\widetilde{\nabla}$ be the connection of G. Take $X, Y \in \mathfrak{p}$.

$$2\langle \widetilde{\nabla}_X Y, Z \rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle + \langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle - \langle [Y, Z], X\rangle$$

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$\mathrm{Exp}(tX)$ is a horizontal G-geodesic $\Rightarrow \mathrm{Exp}(tX) \cdot o$ is an M-geodesic.

Connection and curvature

Proposition

If $X \in \mathfrak{p}$ and $Y \in \Gamma(TM)$, then at o:

$$\nabla_X Y = [X^*, Y].$$

<u>Proof</u>

Define $T_t = L_{\mathrm{Exp}(tX)} = s_{\mathrm{Exp}\left(\frac{tX}{2}\right)\cdot o} s_o$. Then $(T_t)_{*o} = P_o^{\mathrm{Exp}(tX)\cdot o}$.

$$\nabla_X Y = \frac{d}{dt}\Big|_{t=0} (T_{-t})_{*\mathrm{Exp}(tX)\cdot o} Y_{\mathrm{Exp}(tX)\cdot o}$$

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Connection and curvature

Proposition

The curvature tensor $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ at o is

$$R(X,Y)Z = -\big[[X,Y],Z\big].$$

Theorem

Totally geodesic $\Sigma \subseteq M$ through o.	$\longleftrightarrow$	Lie triple systems $V \subseteq \mathfrak{p}$ $\big[[V,V],V\big] \subseteq V$.
$\cup\!\mid$		$\cup\!\mid$
Flat t.g. $\Sigma \subseteq M$ through o.	$\longleftrightarrow$	Abelian $V \subseteq \mathfrak{p}$.

Type

$M = G/K$ symmetric space, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $B: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ Killing form:

$$B(X, Y) = \mathrm{tr}\big(\mathrm{ad}(X)\mathrm{ad}(Y)\big).$$

M is of:

- **Euclidean type** if $B|_{\mathfrak{p}}$ is zero ($\mathbb{R}^n$,…).
- **Compact type** if $B|_{\mathfrak{p}}$ is negative definite ($\mathbb{S}^n$, $\mathbb{F}P^n$, $\mathrm{Gr}_k(\mathbb{R}^n)$,…).
- **Noncompact type** if $B|_{\mathfrak{p}}$ is positive definite ($\mathbb{F}H^n$,$\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$,…).

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Compact type	Noncompact type
• B negative definite. • G compact, semisimple. • $\sec \geq 0$.	• $B\|_{\mathfrak{k}}$ negative definite. • G noncompact, semisimple. • $\sec \leq 0$.

In both cases, $\mathfrak{g} = \mathfrak{i}(M)$.

Noncompact symmetric spaces

$M = G/K$ of noncompact type, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

$$\langle X, Y \rangle = -B(X, \theta Y), \quad X, Y \in \mathfrak{g}.$$

- $\langle \cdot, \cdot \rangle$ is an inner product on $\mathfrak{g}$.
- $\mathrm{ad}(X)^T = -\mathrm{ad}(\theta X)$.
- $\mathrm{ad}(X)$ is symmetric (skew-symmetric) when $X \in \mathfrak{p}$ ($X \in \mathfrak{k}$).

We normalize the metric on M so that it coincides with $\langle \cdot, \cdot \rangle|_{\mathfrak{p}}$ at o.

Noncompact symmetric spaces

$M = G/K$ of noncompact type.

Theorem

M is a Hadamard manifold.

<u>Proof</u>

$\exp_o : \mathfrak{p} \to M$ is a covering map. Assume $\exp_o(X) = \exp_o(Y)$.

$$\mathrm{Exp}(X) \cdot o = \mathrm{Exp}(Y) \cdot o \quad \Rightarrow \quad \mathrm{Exp}(X) = \mathrm{Exp}(Y)k$$

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$$\mathrm{Exp}(X) \cdot o = \mathrm{Exp}(Y) \cdot o \;\Rightarrow\; e^{\mathrm{ad}(X)} = e^{\mathrm{ad}(Y)} \mathrm{Ad}(k)$$
$$\Rightarrow\; Y - X \in \mathfrak{z}(\mathfrak{g}) = 0.$$

Noncompact symmetric spaces

$M = G/K$ of noncompact type.

Theorem

K is a maximal compact subgroup.

Proof

L compact subgroup, $K \subseteq L$.

L fixes a point $p = g \cdot o \Rightarrow K \subseteq L \subseteq G_p$.

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$K = gKg^{-1} = L$.

Root space decomposition

Fix a maximal abelian $\mathfrak{a} \subseteq \mathfrak{p}$.

- $\mathrm{ad}(\mathfrak{a}) \subseteq \mathfrak{gl}(\mathfrak{g})$ commuting family of symmetric endomorphisms.
- We get the **root space decomposition**:

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid [H, X] = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}. \qquad \text{(Root space)}$$

$$\Delta = \{\lambda \in \mathfrak{a}^* \mid \lambda \neq 0, \mathfrak{g}_\lambda \neq 0\}. \qquad \text{(Roots)}$$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Delta} \mathfrak{g}_\lambda$$

Properties of root spaces

$\mathfrak{a}$ maximal abelian subspace, Δ set of roots.

- $\left[\mathfrak{g}_\lambda, \mathfrak{g}_\mu\right] \subseteq \mathfrak{g}_{\lambda+\mu}$.

$X \in \mathfrak{g}_\lambda, Y \in \mathfrak{g}_\mu, H \in \mathfrak{a},$ $\left[H,[X,Y]\right] = -\left[X,[Y,H]\right] - \left[Y,[H,X]\right]$

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- $\theta\mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$.

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$$[H, \theta X] = \theta[\theta H, X]$$

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- $\mathfrak{g}_0 = \mathfrak{z}_\mathfrak{k}(\mathfrak{a}) \oplus \mathfrak{a}$.

$$\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{k}) \oplus (\mathfrak{g}_0 \cap \mathfrak{p})$$

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$$X \in \mathfrak{g}_0 \cap \mathfrak{p} \Rightarrow \mathfrak{a} + \mathbb{R}X = \mathfrak{a}$$

Positive roots

Notion of positivity $\Rightarrow \Delta = \Delta^+ \sqcup \Delta^-$.

Regular elements: elements of

$$\mathfrak{a} \setminus \bigcup_{\lambda \in \Delta} \ker \lambda$$

Definition

Fix a regular $H_0 \in \mathfrak{a}$. $\lambda \in \Delta$ is

- **Positive** ($\lambda \in \Delta^+$) if $\lambda(H_0) > 0$.
- **Negative** ($\lambda \in \Delta^-$) if $\lambda(H_0) < 0$.

Iwasawa decomposition

$M = G/K$ of noncompact type, $\mathfrak{a} \subseteq \mathfrak{p}$ maximal abelian, Δ^+ positive roots.

$$\mathfrak{n} = \bigoplus_{\lambda \in \Delta^+} \mathfrak{g}_\lambda.$$

$\mathfrak{n}$ nilpotent subalgebra, $\mathfrak{a} \oplus \mathfrak{n}$ solvable, $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$.

Theorem (Iwasawa decomposition, Lie algebra level)

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

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Key example

$$\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$$

$$\mathfrak{k} = \mathfrak{so}(n)$$
$$\mathfrak{a} = \{\text{diagonal matrices of trace zero}\}$$
$$\mathfrak{n} = \{\text{strictly upper triangular matrices}\}$$

Theorem (Iwasawa decomposition, Lie algebra level)

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Proof

$\mathfrak{a} \cap \mathfrak{n} = 0$ by definition.
$\mathfrak{g}_0 = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \oplus \mathfrak{a}$.

Assume $X \in \mathfrak{k} \cap (\mathfrak{a} \oplus \mathfrak{n})$.

$$X = X_{\mathfrak{a}} + \sum_{\lambda \in \Delta^+} X_\lambda$$

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<u>Proof</u>

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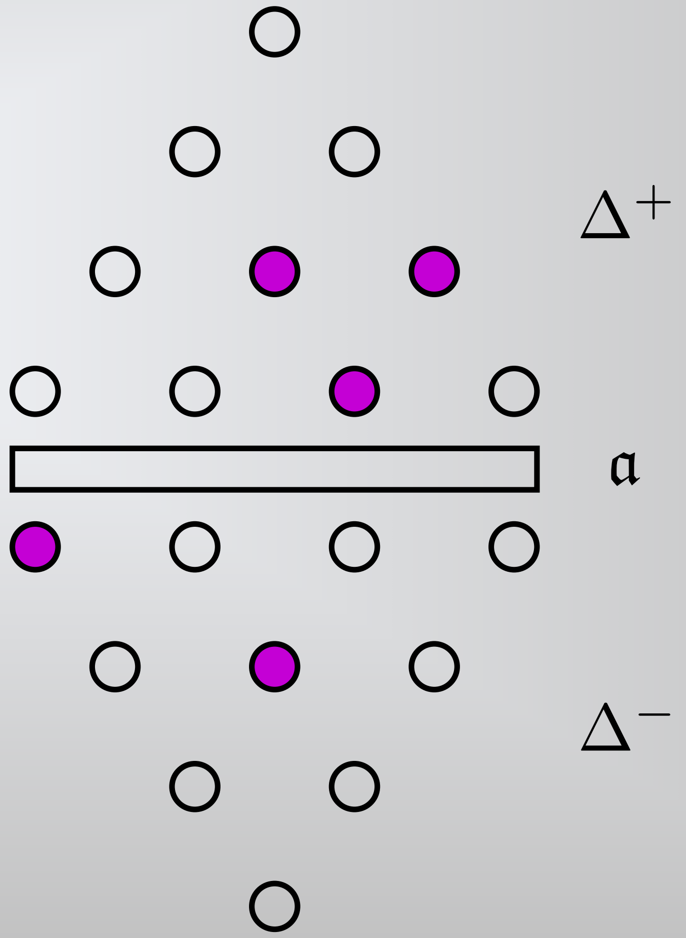
$X \in \mathfrak{g}$

$$X = X_{\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})} + X_{\mathfrak{a}} + \sum_{\lambda \in \Delta^-} X_\lambda + \sum_{\lambda \in \Delta^+} X_\lambda$$



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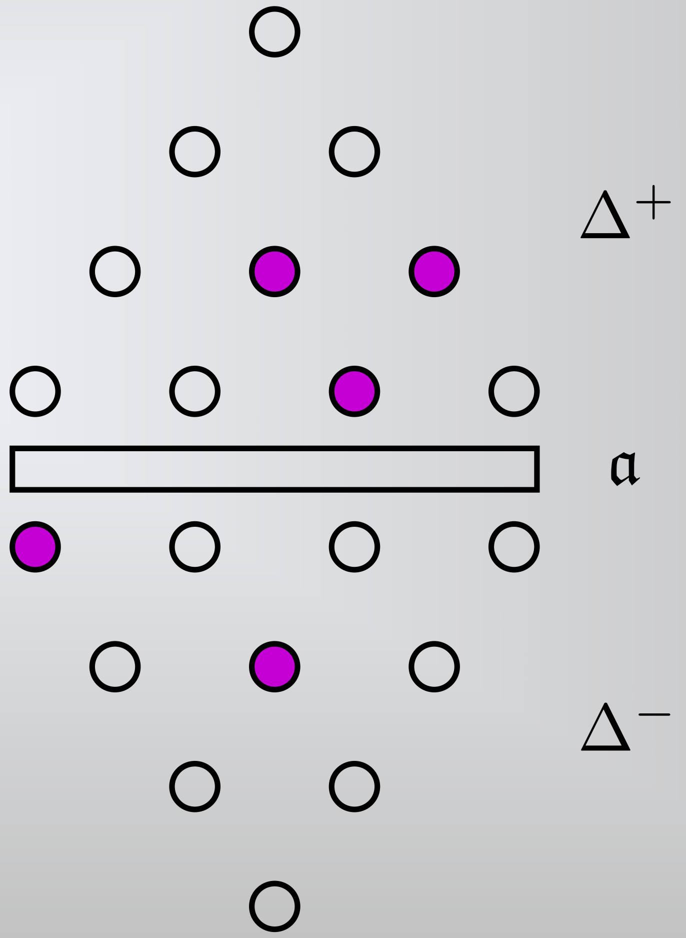

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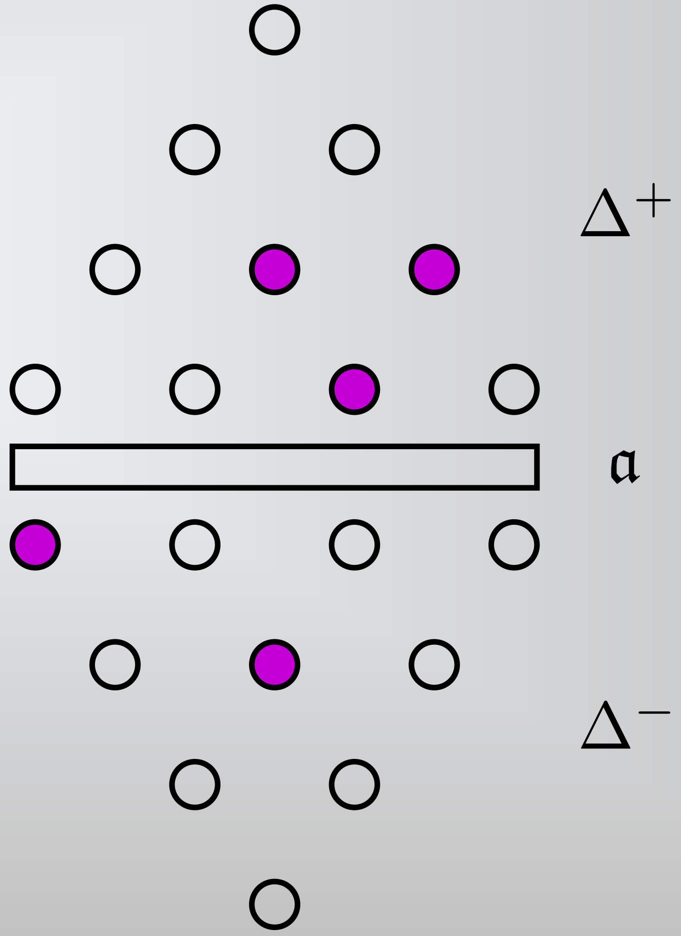

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$$\mathfrak{n} = \bigoplus_{\lambda \in \Delta^+} \mathfrak{g}_\lambda \, .$$

A, N, AN connected subgroups generated by $\mathfrak{a}, \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}$.

Theorem (Iwasawa decomposition, Lie group level)

The multiplication maps $K \times A \times N \to G$ and $A \times N \to AN$ are diffeomorphisms.

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<u>Proof (sketch)</u>

Under a suitable basis:

$$\begin{aligned}\mathrm{ad}(\mathfrak{k}) &\subseteq \mathfrak{so}(\mathfrak{g}),\\ \mathrm{ad}(\mathfrak{a}) &\subseteq \{\text{diagonal matrices}\},\\ \mathrm{ad}(\mathfrak{n}) &\subseteq \{\text{upper triangular matrices with zeros in the diagonal}\}.\end{aligned}$$

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Proof (sketch)

Use Ad to replace G by $G' = \mathrm{Ad}(G) = (\mathrm{Aut}\,\mathfrak{g})^0$.

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<u>Proof (sketch)</u>

Use Ad to replace G by $G' = \mathrm{Ad}(G) = (\mathrm{Aut}\,\mathfrak{g})^0$.

$$\begin{aligned} K' &\subseteq \mathrm{SO}(\mathfrak{g}), \\ A' &\subseteq \{\text{diagonal matrices with positive entries}\}, \\ N' &\subseteq \{\text{upper triangular matrices with ones in the diagonal}\}. \end{aligned}$$

First, prove the result for G'. Then, lift it to G by $\mathrm{Ad}\colon G \to G'$.

The solvable model

$M = G/K$ of noncompact type, $G = KAN$ Iwasawa decomposition.

$$\begin{aligned} \phi: AN &\to M \\ g &\mapsto g \cdot o \end{aligned}$$

is a diffeomorphism. The pullback metric is left-invariant.

Theorem

M is isometric to a simply connected solvable Lie group with a left-invariant metric.

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$M = G/K$ of noncompact type, $G = KAN$ Iwasawa decomposition.

$$\phi: AN \to M$$
$$g \mapsto g \cdot o$$

is a diffeomorphism. The pullback metric is left-invariant.

$$\langle X, Y \rangle_{AN} = \langle X_{\mathfrak{a}}, Y_{\mathfrak{a}} \rangle + \frac{1}{2} \langle X_{\mathfrak{n}}, Y_{\mathfrak{n}} \rangle,$$
$$4\langle \nabla_X Y, Z \rangle_{AN} = \langle [X, Y] + [\theta X, Y] - [X, \theta Y], Z \rangle.$$