Introduction to polar actions

Juan Manuel Lorenzo Naveiro
Universidade de Santiago de Compostela, Spain

Seminar on symmetric spaces, 2021

Setting the stage

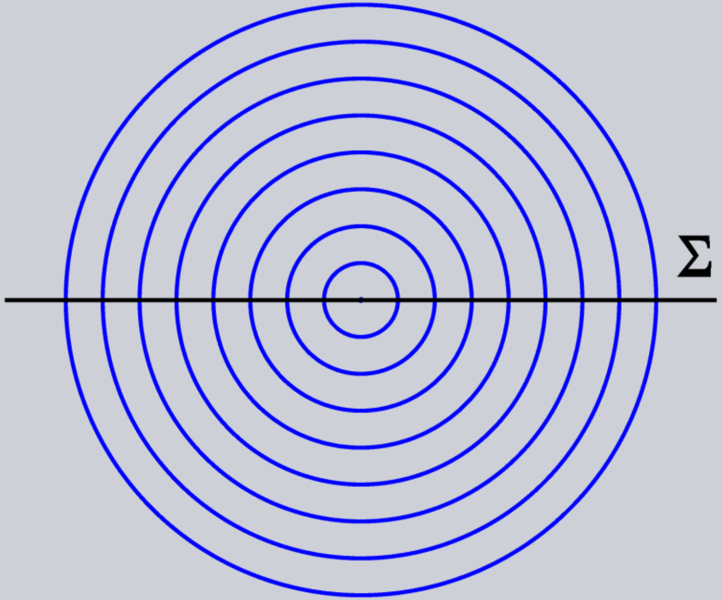
- $(M^n, \langle\cdot,\cdot\rangle)$ complete Riemannian manifold with isometry group $I(M)$.
- $\exp\colon T(M) \to M$ Riemannian exponential map.
- G connected Lie group with Lie algebra $\mathfrak{g}$.
- $\mathrm{Exp}\colon \mathfrak{g} \to G$ Lie exponential map.
- $G \curvearrowright M$ proper isometric action.

$$X_p^* = \frac{d}{dt}\Big|_{t=0} \mathrm{Exp}(tX) \cdot p.$$

Polar actions

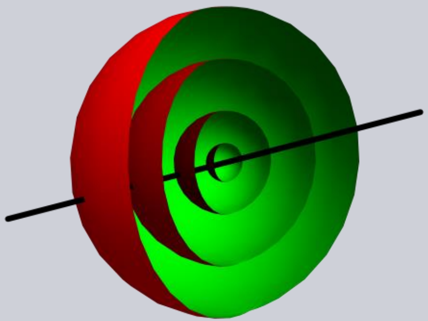
$G \curvearrowright M$ is **polar** if there exists $\Sigma \subseteq M$ such that:

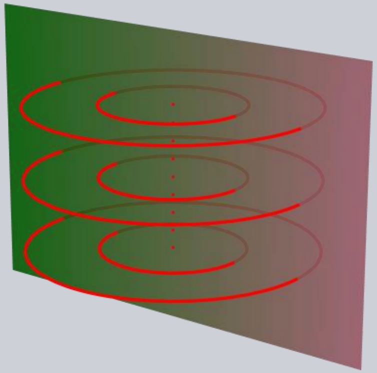
- $\Sigma \cap G \cdot p \neq \emptyset$ for all $p \in M$.
- $T_p\Sigma \perp T_p(G \cdot p)$ for all $p \in \Sigma$.

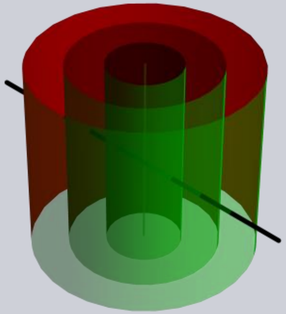


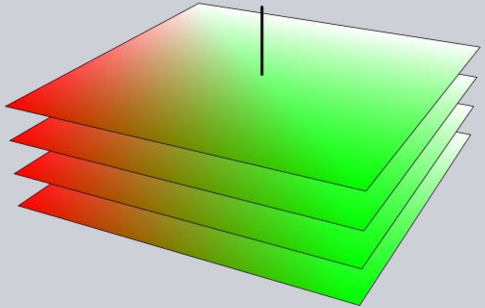
$\mathrm{SO}(2) \curvearrowright \mathbb{R}^2$

Examples in $\mathbb{R}^3$









Some properties

$G \curvearrowright M$ polar action with section Σ.

- $F\colon (g, p) \in G \times \Sigma \mapsto g \cdot p \in M$ is surjective.
- $T_pM = dF_{(e,p)}T_{(e,p)}(G \times \Sigma)$.

Some properties

$G \curvearrowright M$ polar action with section Σ.

- $F\colon (g,p) \in G \times \Sigma \mapsto g \cdot p \in M$ is surjective.
- $T_pM = T_p(G \cdot p) \oplus T_p\Sigma$.
- $G \cdot p$ has maximum dimension (i.e. principal or exceptional).

Proposition

$$\dim(\Sigma) = \min\{\mathrm{codim}(G \cdot p) \mid p \in M\} = \mathrm{cohom}(G \curvearrowright M).$$

Some properties

$G \curvearrowright M$ polar action with section Σ.

- $p \in M$ with $G \cdot p$ principal orbit $\Rightarrow \nu_p\Sigma = T_p(G \cdot p)$.
- Any $\xi \in \nu_p\Sigma$ is of the form

$$\xi = X_p^* = \frac{d}{dt}|_{t=0}\mathrm{Exp}(tX) \cdot p.$$

- $A_\xi : T_p\Sigma \to T_p\Sigma$ is skew-symmetric $\Rightarrow A_\xi = 0$.

Proposition

Σ is a totally geodesic submanifold of M.

Consequences

- If Σ is a section, then $\tilde{\Sigma}$ is a section if and only if $\tilde{\Sigma} = g \cdot \Sigma$ for some $g \in G$.
- If $p \in \Sigma$ belongs to a principal/exceptional orbit, $\Sigma = \exp_p \left(\nu_p(G \cdot p)\right)$.

Question

Given $G \curvearrowright M$ and a regular point $p \in M$, when is $\Sigma = \exp_p \left(\nu_p(G \cdot p)\right)$ a section?

Example: cohomogeneity one actions

$G \curvearrowright M$ cohomogeneity one action.

- $p \in M$ with $G \cdot p$ principal orbit $\Rightarrow G \cdot p$ is a hypersurface.
- $\gamma : \mathbb{R} \to M$ unit speed geodesic with $\gamma'(0) = \xi \in \nu_p(G \cdot p)$.
- $\Sigma = \gamma(\mathbb{R})$ meets all orbits orthogonally: given $X \in \mathfrak{g}$,

Example: cohomogeneity one actions

$G \curvearrowright M$ cohomogeneity one action

- $p \in M$ with $G \cdot p$ principal orbit $\Rightarrow G \cdot p$ is a hypersurface
- $\gamma : \mathbb{R} \to M$ unit speed geodesic with $\gamma'(0) = \xi \in \nu_p(G \cdot p)$
- $\Sigma = \gamma(\mathbb{R})$ meets all orbits orthogonally: given $X \in \mathfrak{g}$,

$$\langle \gamma'(t), X^*_{\gamma(t)} \rangle' = \langle \gamma'(t), \nabla_{\gamma'(t)} X^* \rangle = 0$$

Example: cohomogeneity one actions

$G \curvearrowright M$ cohomogeneity one action

- $p \in M$ with $G \cdot p$ principal orbit $\Rightarrow G \cdot p$ is a hypersurface
- $\gamma\colon \mathbb{R} \to M$ unit speed geodesic with $\gamma'(0) = \xi \in \nu_p(G \cdot p)$
- $\Sigma = \gamma(\mathbb{R})$ meets all orbits orthogonally: given $X \in \mathfrak{g}$,

$$\langle \gamma'(t), X^*_{\gamma(t)} \rangle' = \langle \gamma'(t), \nabla_{\gamma'(t)} X^* \rangle = 0 \Rightarrow \langle \gamma'(t), X^*_{\gamma(t)} \rangle = C$$

Example: cohomogeneity one actions

$G \curvearrowright M$ cohomogeneity one action

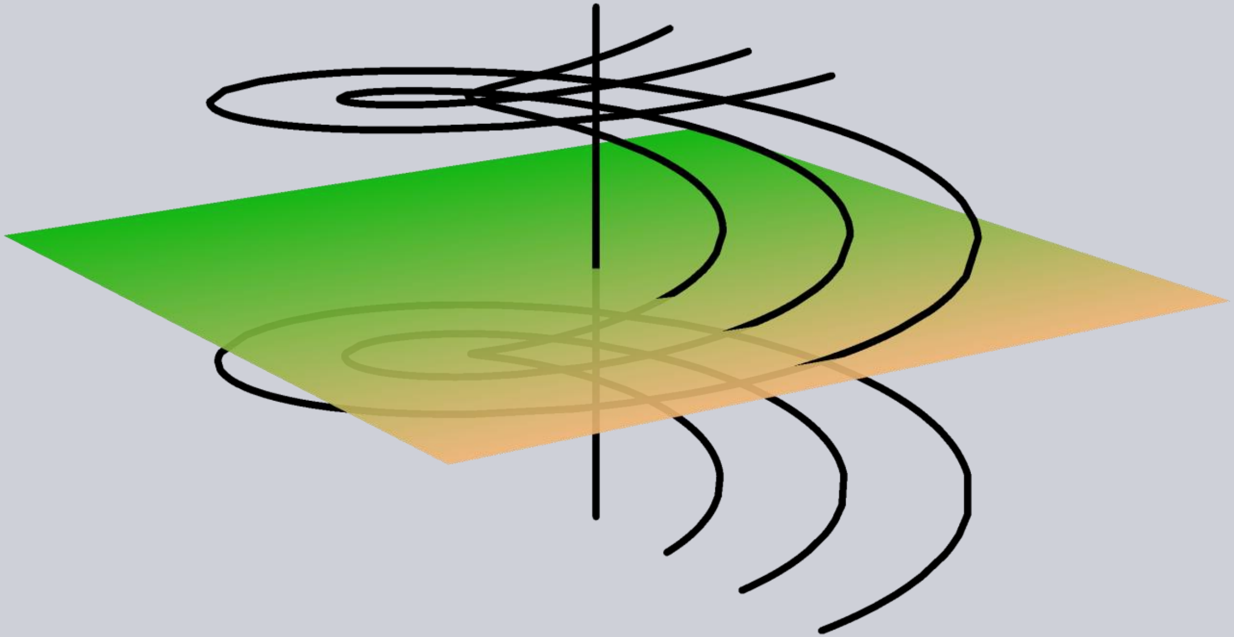
- $p \in M$ with $G \cdot p$ principal orbit $\Rightarrow G \cdot p$ is a hypersurface
- $\gamma: \mathbb{R} \to M$ unit speed geodesic with $\gamma'(0) = \xi \in \nu_p(G \cdot p)$
- $\Sigma = \gamma(\mathbb{R})$ meets all orbits orthogonally: given $X \in \mathfrak{g}$,

$$\langle \gamma'(t), X^*_{\gamma(t)} \rangle' = \langle \gamma'(t), \nabla_{\gamma'(t)} X^* \rangle = 0 \Rightarrow \langle \gamma'(t), X^*_{\gamma(t)} \rangle = 0.$$

Proposition

Every cohomogeneity one action is polar.

False in cohomogeneity two



$$\theta \cdot (z,t) = \left(e^{i\theta}z, t + \theta\right)$$

Example: s-representations

M simply connected semisimple symmetric space, $o \in M$, $G = I^0(M)$, $K = G_o$.

- $\theta: \mathfrak{g} \to \mathfrak{g}$ Cartan involution, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition.

$$\mathfrak{p} \cong T_o M, X \equiv X_o^*.$$

Isotropy representation $K \curvearrowright T_o M \leftrightarrow$ Adjoint representation $K \curvearrowright \mathfrak{p}$ (s-representation).

Theorem

The representation $K \curvearrowright \mathfrak{p}$ is polar. Any maximal abelian $\mathfrak{a} \subseteq \mathfrak{p}$ is a section.

Example: s-representations

M simply connected semisimple symmetric space, $o \in M$, $G = I^0(M)$, $K = G_o$.

- $\theta: \mathfrak{g} \to \mathfrak{g}$ Cartan involution, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition.

$$\mathfrak{p} \cong T_o M, X \equiv X_o^*.$$

Isotropy representation $K \curvearrowright T_o M \leftrightarrow$ Adjoint representation $K \curvearrowright \mathfrak{p}$ (s-representation).

Corollary

Maximal abelian subspaces of $\mathfrak{p}$ are conjugate.

Example: s-representations

M simply connected semisimple symmetric space, $o \in M$, $G = I^0(M)$, $K = G_o$.

- $\theta: \mathfrak{g} \to \mathfrak{g}$ Cartan involution, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition.

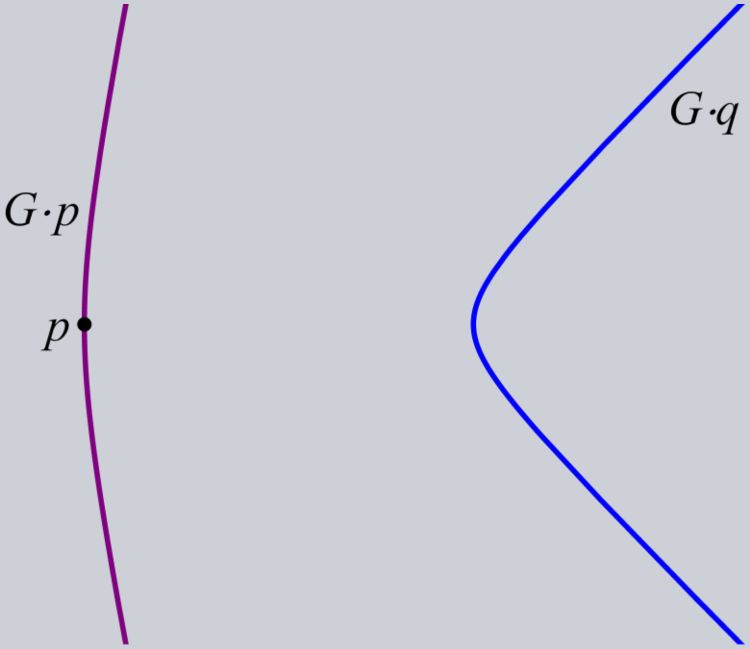
$$\mathfrak{p} \cong T_o M, X \equiv X_o^*.$$

Isotropy representation $K \curvearrowright T_o M$ ↔ Adjoint representation $K \curvearrowright \mathfrak{p}$ (s-representation).

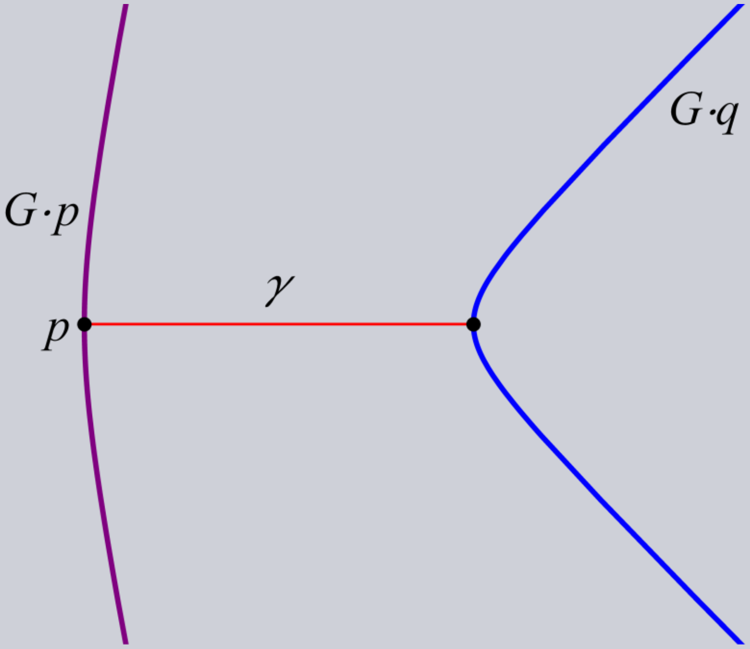
Theorem (Dadok)

Every polar representation in $\mathbb{R}^n$ is orbit equivalent to an s-representation.

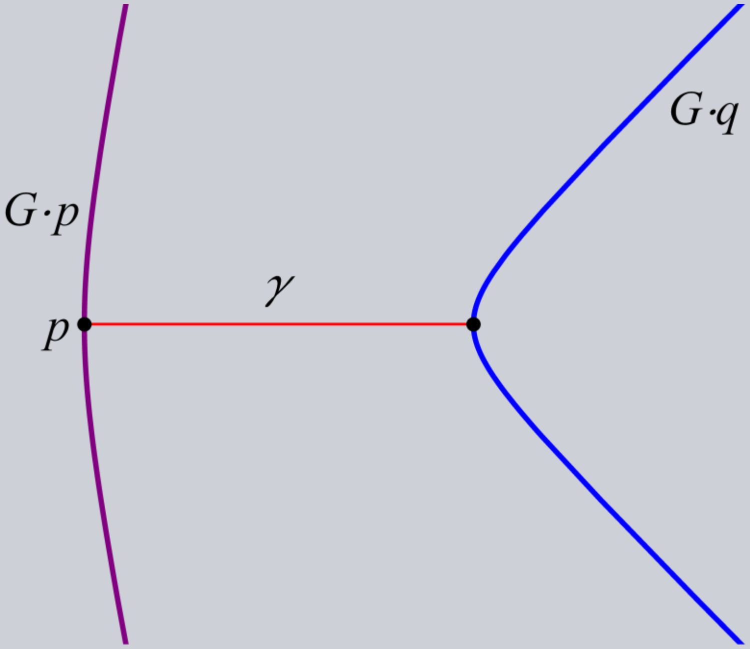
The set $\Sigma = \exp_p\left(\nu_p(G \cdot p)\right)$ intersects all orbits



The set $\Sigma = \exp_p\left(\nu_p(G \cdot p)\right)$ intersects all orbits



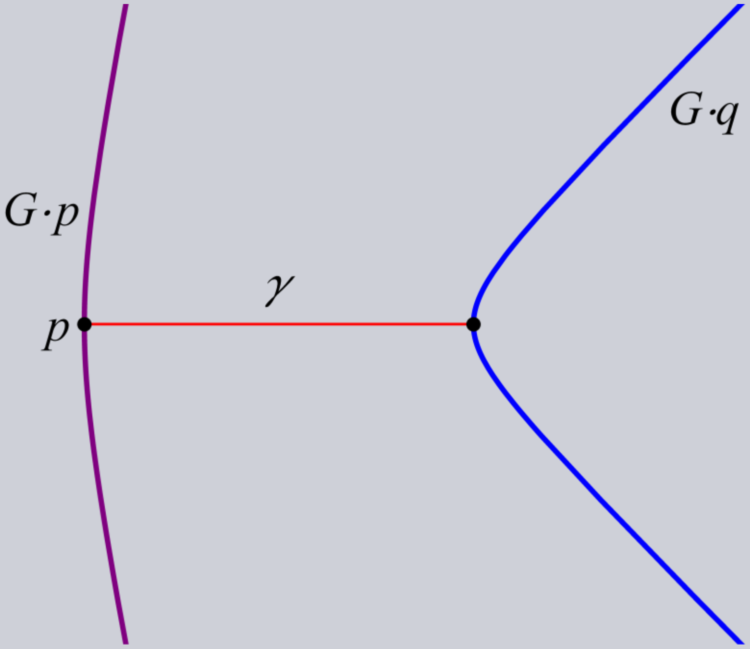
The set $\Sigma = \exp_p\left(\nu_p(G \cdot p)\right)$ intersects all orbits



$$\gamma(t) = \exp_p(t\xi),$$

$$L'(0) = 0$$

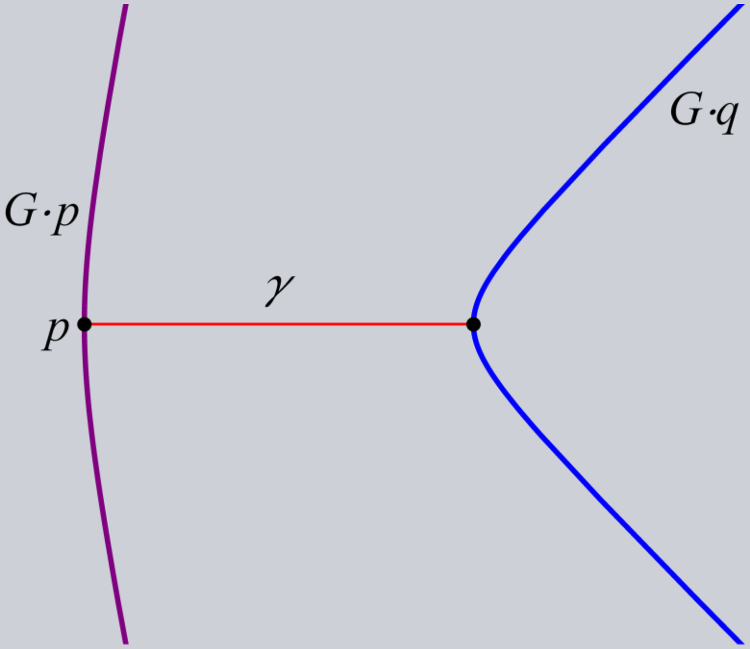
The set $\Sigma = \exp_p\left(\nu_p(G \cdot p)\right)$ intersects all orbits



$$\gamma(t) = \exp_p(t\xi),$$

$$L'(0) = -\int_0^1 \langle V, \gamma'' \rangle dt$$
$$+\langle V(1), \gamma'(1) \rangle - \langle V(0), \gamma'(0) \rangle$$

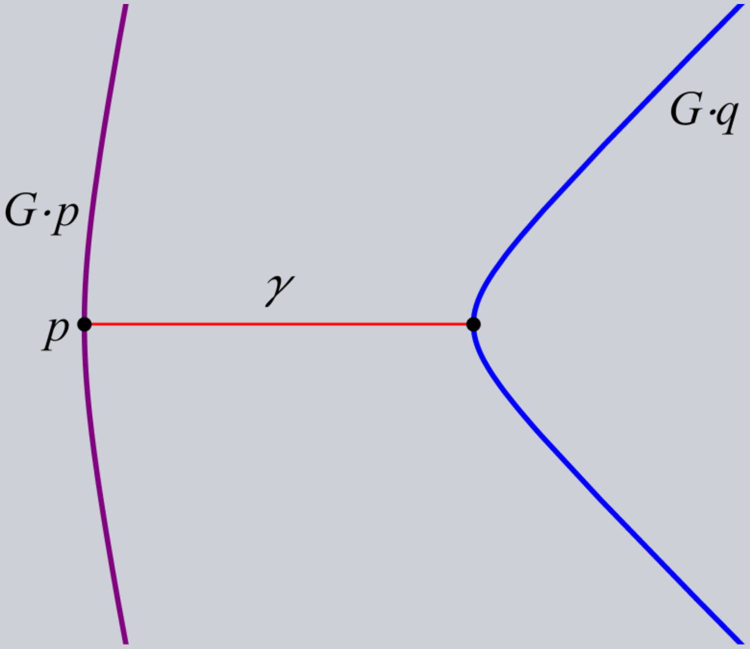
The set $\Sigma = \exp_p\left(\nu_p(G \cdot p)\right)$ intersects all orbits



$$\gamma(t) = \exp_p(t\xi),$$

$$L'(0) = -\langle V(0), \xi \rangle$$

The set $\Sigma = \exp_p\left(\nu_p(G \cdot p)\right)$ intersects all orbits



$$\gamma(t) = \exp_p(t\xi),$$

$$L'(0) = -\langle V(0), \xi\rangle$$
$$\Downarrow$$
$$\xi \in \nu_p(G \cdot p)$$

Orthogonality

- $\Sigma \subseteq M$ complete totally geodesic submanifold, $X \in \mathfrak{X}(M)$ Killing vector field.
- $\gamma: \mathbb{R} \to \Sigma$ geodesic, $J(t) = X_{\gamma(t)}$.

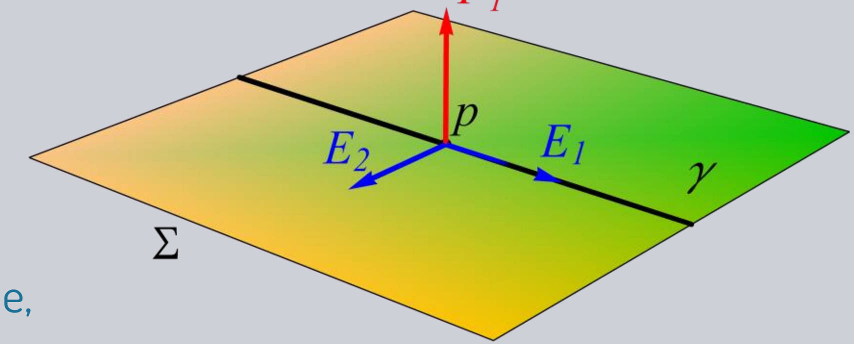
$$J'' + R(J, \gamma')\gamma' = 0$$

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Orthogonality

- $\Sigma \subseteq M$ complete totally geodesic submanifold, $X \in \mathfrak{X}(M)$ Killing vector field.
- $\gamma : \mathbb{R} \to \Sigma$ geodesic, $J(t) = X_{\gamma(t)}$.
- $\{E_i, F_j\}$ adapted parallel ortonormal frame, $E_1 = \gamma'$.

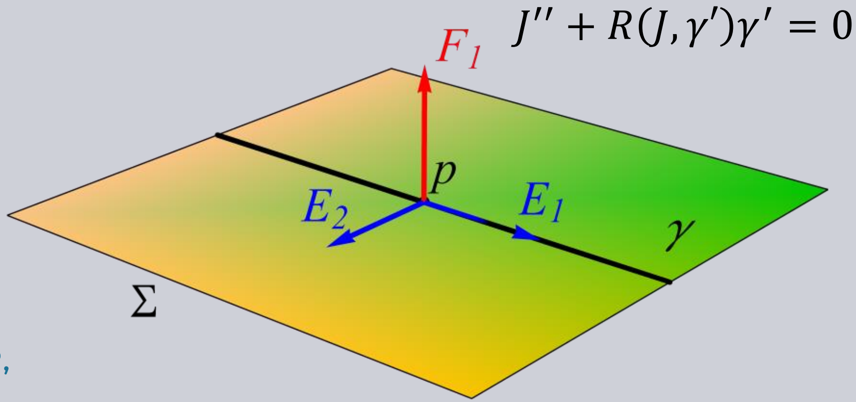
$$J'' + R(J,\gamma')\gamma' = 0$$



$$J = a^i E_i + b^j F_j$$

Orthogonality

- $\Sigma \subseteq M$ complete totally geodesic submanifold, $X \in \mathfrak{X}(M)$ Killing vector field.
- $\gamma : \mathbb{R} \to \Sigma$ geodesic, $J(t) = X_{\gamma(t)}$.
- $\{E_i, F_j\}$ adapted parallel ortonormal frame, $E_1 = \gamma'$.



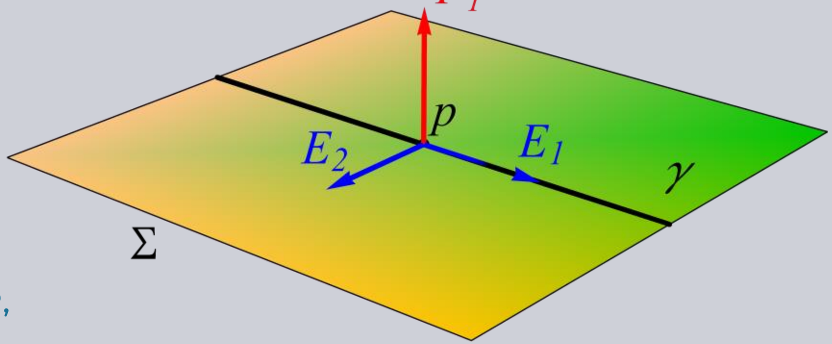
$$J = a^i E_i + b^j F_j$$

$$R(E_i, \gamma')\gamma' = c^{ik} E_k$$
$$\langle R(F_j, \gamma')\gamma', E_l\rangle = -\langle R(E_1, E_l)E_1, F_j\rangle$$

Orthogonality

- $\Sigma \subseteq M$ complete totally geodesic submanifold, $X \in \mathfrak{X}(M)$ Killing vector field.
- $\gamma: \mathbb{R} \to \Sigma$ geodesic, $J(t) = X_{\gamma(t)}$.
- $\{E_i, F_j\}$ adapted parallel ortonormal frame, $E_1 = \gamma'$.

$$J'' + R(J, \gamma')\gamma' = 0$$

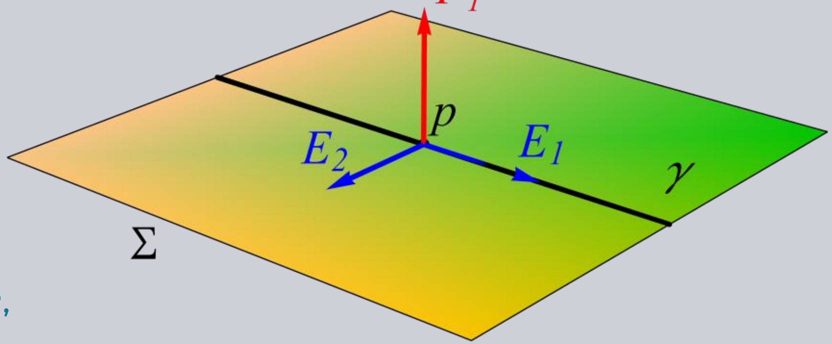


$$J = a^i E_i + b^j F_j$$

$$R(E_i, \gamma')\gamma' = c^{ik} E_k$$
$$\langle R(F_j, \gamma')\gamma', E_l \rangle = 0$$

Orthogonality

- $\Sigma \subseteq M$ complete totally geodesic submanifold, $X \in \mathfrak{X}(M)$ Killing vector field.
- $\gamma \colon \mathbb{R} \to \Sigma$ geodesic, $J(t) = X_{\gamma(t)}$.
- $\{E_i, F_j\}$ adapted parallel ortonormal frame, $E_1 = \gamma'$.



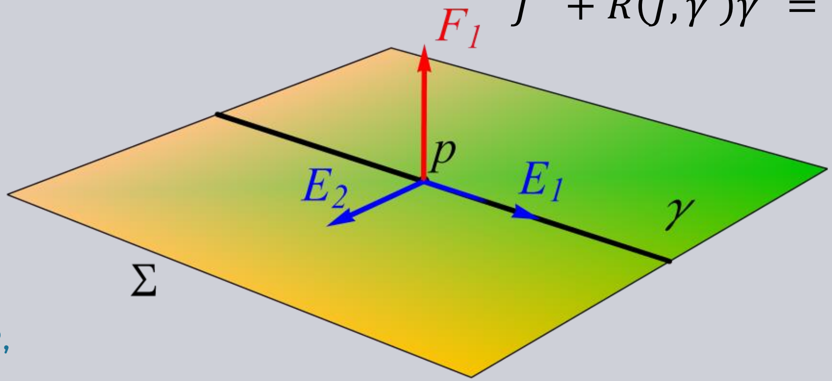
$$J = a^i E_i + b^j F_j$$

$$(a^i)'' E_i + a^k c^{ki} E_i + (b^j)'' F_j + b^j R(F_j, \gamma')\gamma' = 0$$

Orthogonality

- $\Sigma \subseteq M$ complete totally geodesic submanifold, $X \in \mathfrak{X}(M)$ Killing vector field.
- $\gamma: \mathbb{R} \to \Sigma$ geodesic, $J(t) = X_{\gamma(t)}$.
- $\{E_i, F_j\}$ adapted parallel ortonormal frame, $E_1 = \gamma'$.

$$J'' + R(J, \gamma')\gamma' = 0$$



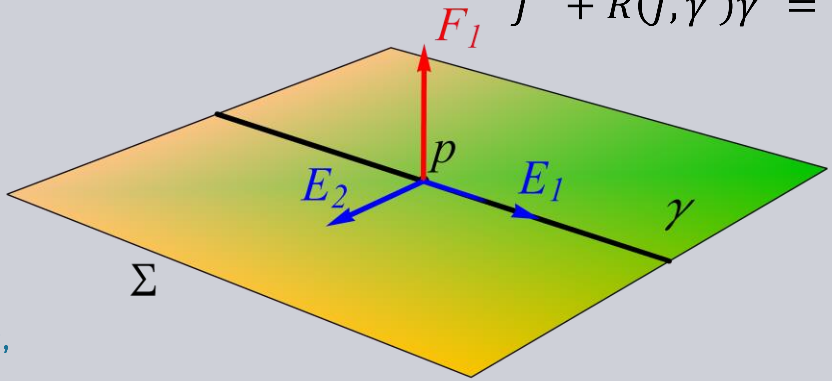
$$J = a^i E_i + b^j F_j$$

$$(a^i)'' + a^k c^{ki} = 0$$

Orthogonality

- $\Sigma \subseteq M$ complete totally geodesic submanifold, $X \in \mathfrak{X}(M)$ Killing vector field.
- $\gamma : \mathbb{R} \to \Sigma$ geodesic, $J(t) = X_{\gamma(t)}$.
- $\{E_i, F_j\}$ adapted parallel ortonormal frame, $E_1 = \gamma'$.

$$J'' + R(J, \gamma')\gamma' = 0$$

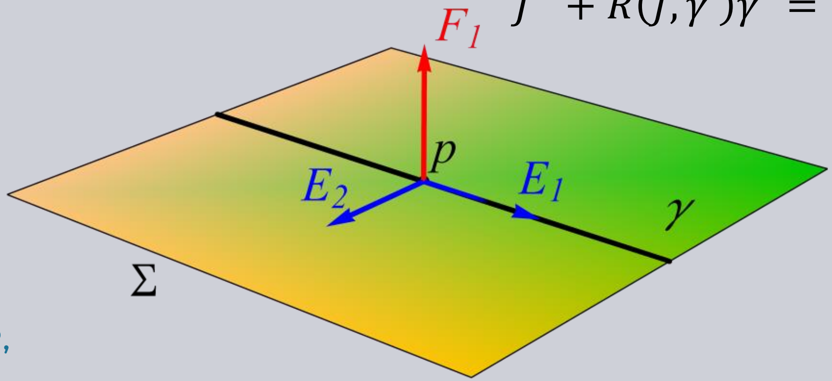


$$J = a^i E_i + b^j F_j$$

$$J \perp \Sigma \Leftrightarrow J(0), J'(0) \perp T_p\Sigma$$

Orthogonality

- $\Sigma \subseteq M$ complete totally geodesic submanifold, $X \in \mathfrak{X}(M)$ Killing vector field.
- $\gamma: \mathbb{R} \to \Sigma$ geodesic, $J(t) = X_{\gamma(t)}$.
- $\{E_i, F_j\}$ adapted parallel ortonormal frame, $E_1 = \gamma'$.



$$J'' + R(J, \gamma')\gamma' = 0$$

$$J = a^i E_i + b^j F_j$$

$$J \perp \Sigma \Leftrightarrow X_p, \nabla_{\gamma'(0)} X \perp T_p\Sigma$$

Orthogonality

- $\Sigma \subseteq M$ complete totally geodesic submanifold, $X \in \mathfrak{X}(M)$ Killing vector field, $p \in M$.

Theorem

X is orthogonal to Σ iff $X_p \in \nu_p\Sigma$ and $\nabla_\xi X \in \nu_p\Sigma$ for all $\xi \in T_p\Sigma$.

Corollary

Given an isometric action $G \curvearrowright M$ and a totally geodesic $\Sigma \subseteq M$, Σ is orthogonal to all the orbits it meets iff $X_p^* \in \nu_p\Sigma$ and $\nabla_\xi X^* \in \nu_p\Sigma$ for all $\xi \in T_p\Sigma$ and $X \in \mathfrak{g}$.

T.g. submanifolds of symmetric spaces

- $M = G/K$ symmetric space, $G = I^0(M)$, $K = G_o$.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition, θ Cartan involution.

$$\nabla_{X^*}T = [X^*, T]$$
$$R(X^*, Y^*)Z^* = -\big[[X, Y], Z\big]^*$$
$$\exp_o(tX) = \mathrm{Exp}(tX) \cdot o$$

T.g. submanifolds of symmetric spaces

- $M = G/K$ symmetric space, $G = I^0(M)$, $K = G_o$.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition, θ Cartan involution.

$$\nabla_{X^*} T = [X^*, T]$$
$$R(X^*, Y^*)Z^* = -[[X, Y], Z]^*$$
$$\exp_o(tX) = \mathrm{Exp}(tX) \cdot o$$

$\Sigma \subseteq M$ totally geodesic, $o \in \Sigma \Rightarrow V = T_o\Sigma \subseteq \mathfrak{p}$ satisfies $[[V, V], V] \subseteq V$

T.g. submanifolds of symmetric spaces

- $M = G/K$ symmetric space, $G = I^0(M)$, $K = G_o$.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition, θ Cartan involution.

$$\nabla_{X^*} T = [X^*, T]$$
$$R(X^*, Y^*)Z^* = -\big[[X, Y], Z\big]^*$$
$$\exp_o(tX) = \operatorname{Exp}(tX) \cdot o$$

Assume $V \subseteq \mathfrak{p}$ satisfies $\big[[V, V], V\big] \subseteq V$

$$\mathfrak{b} = \mathrm{Lie}(V)$$

T.g. submanifolds of symmetric spaces

- $M = G/K$ symmetric space, $G = I^0(M)$, $K = G_o$.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition, θ Cartan involution.

$$\nabla_{X^*} T = [X^*, T]$$
$$R(X^*, Y^*)Z^* = -[[X, Y], Z]^*$$
$$\exp_o(tX) = \mathrm{Exp}(tX) \cdot o$$

Assume $V \subseteq \mathfrak{p}$ satisfies $[[V, V], V] \subseteq V$

$\mathfrak{b} = [V, V] \oplus V$

T.g. submanifolds of symmetric spaces

- $M = G/K$ symmetric space, $G = I^0(M)$, $K = G_o$.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition, θ Cartan involution.

$$\nabla_{X^*} T = [X^*, T]$$
$$R(X^*, Y^*)Z^* = -\big[[X, Y], Z\big]^*$$
$$\exp_o(tX) = \mathrm{Exp}(tX) \cdot o$$

Assume $V \subseteq \mathfrak{p}$ satisfies $\big[[V, V], V\big] \subseteq V$

$\mathfrak{b} = [V, V] \oplus V \rightsquigarrow B \subseteq G$ Lie subgroup

T.g. submanifolds of symmetric spaces

- $M = G/K$ symmetric space, $G = I^0(M)$, $K = G_o$.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition, θ Cartan involution.

$$\nabla_{X^*}T = [X^*, T]$$
$$R(X^*, Y^*)Z^* = -\big[[X, Y], Z\big]^*$$
$$\exp_o(tX) = \mathrm{Exp}(tX) \cdot o$$

Assume $V \subseteq \mathfrak{p}$ satisfies $\big[[V, V], V\big] \subseteq V$

$\mathfrak{b} = [V, V] \oplus V \rightsquigarrow B \subseteq G$ Lie subgroup

$\Sigma = B \cdot o$ complete, totally geodesic with tangent space $\mathfrak{b}_{\mathfrak{p}} = V$

T.g. submanifolds of symmetric spaces

- $M = G/K$ symmetric space, $G = I^0(M)$, $K = G_o$.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition, θ Cartan involution.

$$\nabla_{X^*}T = [X^*, T]$$
$$R(X^*, Y^*)Z^* = -\big[[X, Y], Z\big]^*$$
$$\exp_o(tX) = \mathrm{Exp}(tX) \cdot o$$

Theorem

There is a bijective correspondence

Complete totally geodesic submanifolds through o. $\leftrightarrow$ Lie triple systems in $\mathfrak{p}$

T.g. submanifolds of symmetric spaces

- $M = G/K$ symmetric space, $G = I^0(M)$, $K = G_o$.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition, θ Cartan involution.

$$\nabla_{X^*} T = [X^*, T]$$
$$R(X^*, Y^*)Z^* = -\big[[X, Y], Z\big]^*$$
$$\exp_o(tX) = \mathrm{Exp}(tX) \cdot o$$

Theorem

There is a bijective correspondence

Complete flat totally geodesic submanifolds through o. $\leftrightarrow$ Abelian subspaces of $\mathfrak{p}$

Polarity criterion

$M = G/K$ of compact type

- $\langle\cdot,\cdot\rangle$ $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$.
- $\mathrm{ad}(X)$ skew-symmetric for all $X \in \mathfrak{g}$.
- Extend $\langle\cdot,\cdot\rangle_{\mathfrak{p}}$ to a G-invariant metric on M.

$M = G/K$ of noncompact type

- $\langle X, Y\rangle = -B(X, \theta Y)$ is an inner product on $\mathfrak{g}$.
- $\mathrm{ad}(X)$ skew-symmetric for all $X \in \mathfrak{k}$.
- $\mathrm{ad}(X)$ symmetric for all $X \in \mathfrak{p}$.
- Extend $\langle\cdot,\cdot\rangle_{\mathfrak{p}}$ to a G-invariant metric on M.

Polarity criterion

$H \leq G$ connected closed subgroup, $H \cdot o$ principal orbit.

Question

When is $H \curvearrowright M$ a polar action?

Polarity criterion

$H \leq G$ connected closed subgroup, $H \cdot o$ principal orbit.

$$\mathfrak{h}_{\mathfrak{p}}^{\perp} = \{X \in \mathfrak{p} \mid \langle X, Y \rangle = 0 \text{ for all } Y \in \mathfrak{h}\} = \mathfrak{p} \ominus \mathfrak{h}_{\mathfrak{p}}.$$

Polarity criterion

$H \leq G$ connected closed subgroup, $H \cdot o$ principal orbit.

$$\mathfrak{h}_{\mathfrak{p}}^{\perp} = \{X \in \mathfrak{p} \mid \langle X, Y \rangle = 0 \text{ for all } Y \in \mathfrak{h}\} = \mathfrak{p} \ominus \mathfrak{h}_{\mathfrak{p}}.$$

$$H \curvearrowright M \text{ polar} \Leftrightarrow \Sigma = \exp_o\left(\mathfrak{h}_{\mathfrak{p}}^{\perp}\right) \text{ is a section}$$

- $\mathfrak{h}_{\mathfrak{p}}^{\perp}$ is a Lie Triple System.
- $X_o^* \in \nu_o\Sigma$ and $\nabla_\xi X^* \in \nu_o\Sigma$ for all $X \in \mathfrak{h}$ and $\xi \in \mathfrak{h}_{\mathfrak{p}}^{\perp}$.

$X \in \mathfrak{h},\ \ \xi, \eta \in \mathfrak{h}_{\mathfrak{p}}^{\perp}$ $\qquad$ $0 = \langle \nabla_\xi X^*, \eta \rangle$

Polarity criterion

$H \leq G$ connected closed subgroup, $H \cdot o$ principal orbit.

$$\mathfrak{h}_{\mathfrak{p}}^{\perp} = \{X \in \mathfrak{p} \mid \langle X, Y \rangle = 0 \text{ for all } Y \in \mathfrak{h}\} = \mathfrak{p} \ominus \mathfrak{h}_{\mathfrak{p}}.$$

$$H \curvearrowright M \text{ polar} \Leftrightarrow \Sigma = \exp_o(\mathfrak{h}_{\mathfrak{p}}^{\perp}) \text{ is a section}$$

- $\mathfrak{h}_{\mathfrak{p}}^{\perp}$ is a Lie Triple System.
- $X_o^* \in \nu_o\Sigma$ and $\nabla_\xi X^* \in \nu_o\Sigma$ for all $X \in \mathfrak{h}$ and $\xi \in \mathfrak{h}_{\mathfrak{p}}^{\perp}$.

$X \in \mathfrak{h}, \ \xi, \eta \in \mathfrak{h}_{\mathfrak{p}}^{\perp}$ $\qquad$ $0 = \langle [\xi^*, X^*], \eta \rangle$

Polarity criterion

$H \leq G$ connected closed subgroup, $H \cdot o$ principal orbit.

$$\mathfrak{h}_{\mathfrak{p}}^{\perp} = \{X \in \mathfrak{p} \mid \langle X, Y \rangle = 0 \text{ for all } Y \in \mathfrak{h}\} = \mathfrak{p} \ominus \mathfrak{h}_{\mathfrak{p}}.$$

$$H \curvearrowright M \text{ polar} \Leftrightarrow \Sigma = \exp_o\left(\mathfrak{h}_{\mathfrak{p}}^{\perp}\right) \text{ is a section}$$

- $\mathfrak{h}_{\mathfrak{p}}^{\perp}$ is a Lie Triple System.
- $X_o^* \in \nu_o\Sigma$ and $\nabla_\xi X^* \in \nu_o\Sigma$ for all $X \in \mathfrak{h}$ and $\xi \in \mathfrak{h}_{\mathfrak{p}}^{\perp}$.

$X \in \mathfrak{h},\ \ \xi, \eta \in \mathfrak{h}_{\mathfrak{p}}^{\perp}$

$0 = \langle -[\xi, X]^*, \eta \rangle$

Polarity criterion

$H \leq G$ connected closed subgroup, $H \cdot o$ principal orbit.

$$\mathfrak{h}_{\mathfrak{p}}^{\perp} = \{X \in \mathfrak{p} \mid \langle X, Y \rangle = 0 \text{ for all } Y \in \mathfrak{h}\} = \mathfrak{p} \ominus \mathfrak{h}_{\mathfrak{p}}.$$

$$H \curvearrowright M \text{ polar} \Leftrightarrow \Sigma = \exp_o\left(\mathfrak{h}_{\mathfrak{p}}^{\perp}\right) \text{ is a section}$$

- $\mathfrak{h}_{\mathfrak{p}}^{\perp}$ is a Lie Triple System.
- $X_o^* \in \nu_o \Sigma$ and $\nabla_\xi X^* \in \nu_o \Sigma$ for all $X \in \mathfrak{h}$ and $\xi \in \mathfrak{h}_{\mathfrak{p}}^{\perp}$.

$X \in \mathfrak{h}, \ \xi, \eta \in \mathfrak{h}_{\mathfrak{p}}^{\perp}$ $\qquad$ $0 = \langle -[\xi, X], \eta \rangle$

Polarity criterion

$H \leq G$ connected closed subgroup, $H \cdot o$ principal orbit.

$$\mathfrak{h}_{\mathfrak{p}}^{\perp} = \{X \in \mathfrak{p} \mid \langle X, Y \rangle = 0 \text{ for all } Y \in \mathfrak{h}\} = \mathfrak{p} \ominus \mathfrak{h}_{\mathfrak{p}}.$$

$$H \curvearrowright M \text{ polar} \Leftrightarrow \Sigma = \exp_o\left(\mathfrak{h}_{\mathfrak{p}}^{\perp}\right) \text{ is a section}$$

- $\mathfrak{h}_{\mathfrak{p}}^{\perp}$ is a Lie Triple System.
- $X_o^* \in \nu_o\Sigma$ and $\nabla_\xi X^* \in \nu_o\Sigma$ for all $X \in \mathfrak{h}$ and $\xi \in \mathfrak{h}_{\mathfrak{p}}^{\perp}$.

$X \in \mathfrak{h},\ \ \xi, \eta \in \mathfrak{h}_{\mathfrak{p}}^{\perp}$ $\qquad\qquad 0 = \pm\langle X, [\xi, \eta]\rangle$

Polarity criterion

$H \leq G$ connected closed subgroup, $H \cdot o$ principal orbit.

$$\mathfrak{h}_{\mathfrak{p}}^{\perp} = \{X \in \mathfrak{p} \mid \langle X, Y \rangle = 0 \text{ for all } Y \in \mathfrak{h}\} = \mathfrak{p} \ominus \mathfrak{h}_{\mathfrak{p}}.$$

$$H \curvearrowright M \text{ polar} \Leftrightarrow \Sigma = \exp_o\left(\mathfrak{h}_{\mathfrak{p}}^{\perp}\right) \text{ is a section}$$

Theorem (Gorodski/Berndt, Díaz-Ramos, Tamaru)

$H \curvearrowright M$ is polar if and only if $\mathfrak{h}_{\mathfrak{p}}^{\perp}$ is a Lie Triple System and $\left[\mathfrak{h}_{\mathfrak{p}}^{\perp}, \mathfrak{h}_{\mathfrak{p}}^{\perp}\right] \perp \mathfrak{h}$.
The action is hyperpolar if and only if $\left[\mathfrak{h}_{\mathfrak{p}}^{\perp}, \mathfrak{h}_{\mathfrak{p}}^{\perp}\right] = 0$.


LOONEY TUNES
"That's all Folks!"