Polar homogeneous foliations on symmetric spaces of noncompact type

Juan Manuel Lorenzo Naveiro

Universidade de Santiago de Compostela

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GALICIAN CENTRE FOR MATHEMATICAL RESEARCH AND TECHNOLOGY

USC UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

XUNTA DE GALICIA

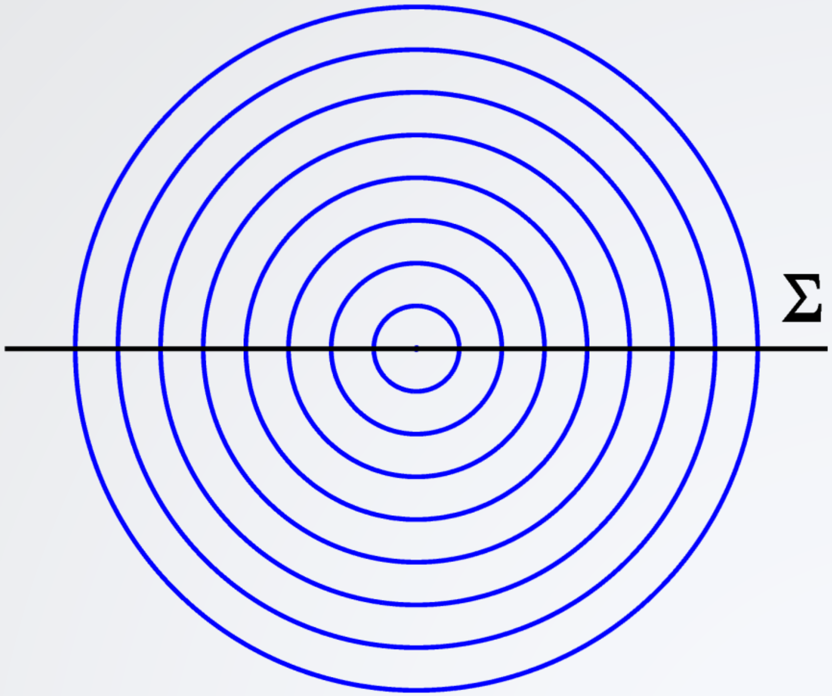
GOBIERNO DE ESPAÑA

MINISTERIO DE CIENCIA E INNOVACIÓN

Polar actions

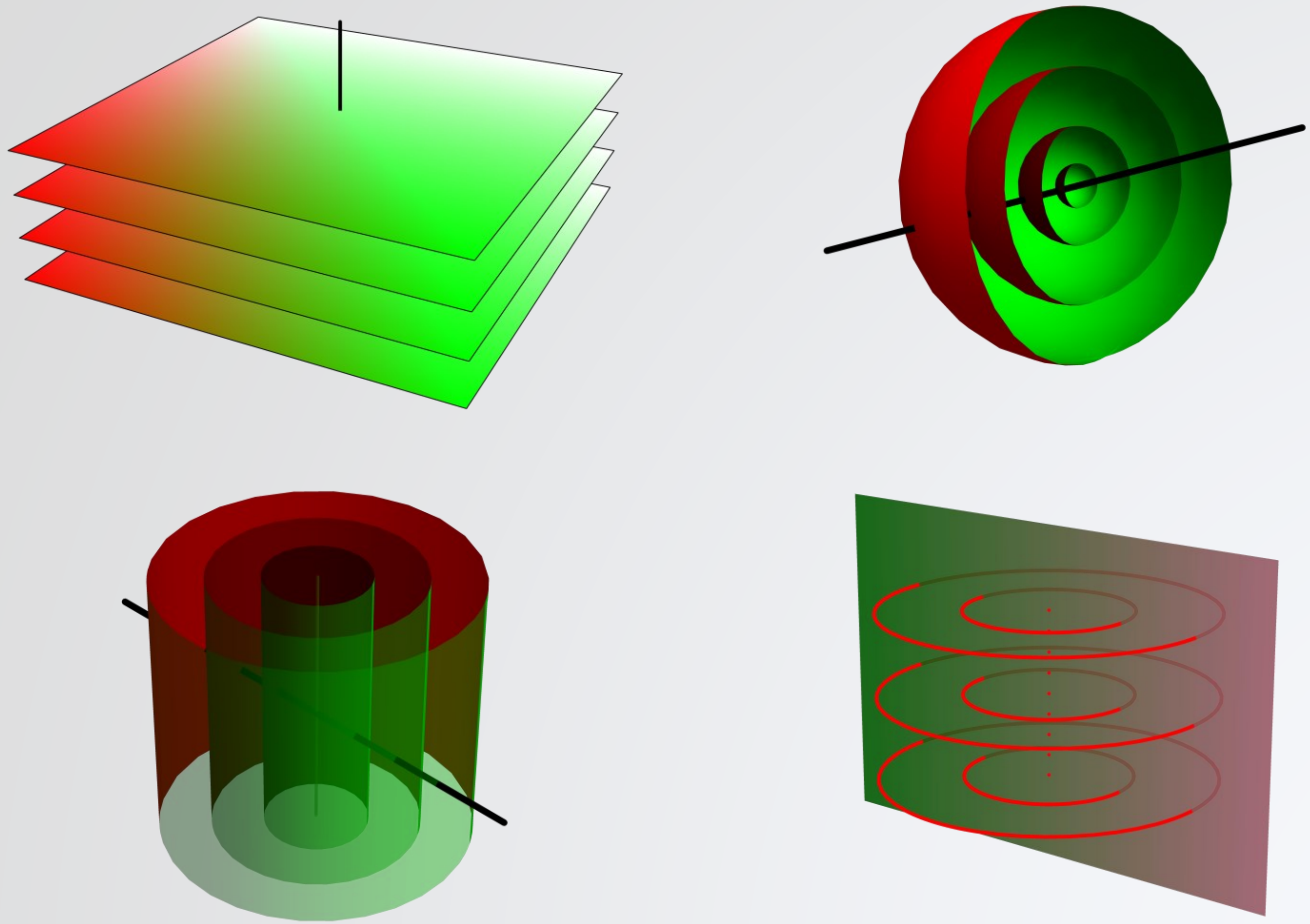
- M complete Riemannian manifold, $G \curvearrowright M$ proper isometric action.

$G \curvearrowright M$ is *polar* if there exists $\Sigma \subseteq M$ intersecting all orbits orthogonally.



- $\dim(\Sigma) = \mathrm{cohom}(G \curvearrowright M)$.
- Σ is totally geodesic.

Examples



Noncompact symmetric spaces

- $M = G/K$ of noncompact type, $G = I^0(M), K = G_o$.
- $\theta: \mathfrak{g} \to \mathfrak{g}$ Cartan involution.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition, $\mathfrak{p} = T_o M$.
- $\langle X, Y \rangle = -B(X, \theta Y)$ inner product.
- Totally geodesic submanifolds Lie triple systems in $\mathfrak{p}$.

$$V \subseteq \mathfrak{p}, \left[V, [V, V]\right] \subseteq V.$$

Root spaces

- $\mathfrak{a} \subseteq \mathfrak{p}$ maximal abelian subspace, $\alpha \in \mathfrak{a}^*$.

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}.$$

- Root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \right)$$

- Σ^+ positive roots, Λ simple roots.
- Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \qquad \mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$$

Split Lie algebras

- $\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{k}) \oplus (\mathfrak{g}_0 \cap \mathfrak{p}) = \mathfrak{k}_0 \oplus \mathfrak{a}$.
- $\mathfrak{g}$ is *split* if one of the following equivalent conditions is satisfied:
  a) $\mathfrak{k}_0 = 0$.

  b) $\mathfrak{g}_0 = \mathfrak{a}$.

  c) $\mathfrak{a}$ is self-normalizing, $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{a}$.
- Examples: $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{so}(n, n+1)$, $\mathfrak{so}(n, n)$, $\mathfrak{sp}(n, \mathbb{R})\ldots$

Algebraic characterization

- $H \leq G$ closed connected subgroup inducing a homogeneous foliation.
- Consider the normal space

$$\mathfrak{h}_{\mathfrak{p}}^{\perp} = \{X \in \mathfrak{p} : \langle X, \mathfrak{h} \rangle = 0\} = \mathfrak{p} \ominus \mathfrak{h}_{\mathfrak{p}}.$$

Theorem

- $H \curvearrowright M$ is polar $\Leftrightarrow$ $\mathfrak{h}_{\mathfrak{p}}^{\perp}$ is a Lie triple system and $\left[\mathfrak{h}_{\mathfrak{p}}^{\perp}, \mathfrak{h}_{\mathfrak{p}}^{\perp}\right] \perp \mathfrak{h}$.
- $H \curvearrowright M$ is hyperpolar $\Leftrightarrow$ $\left[\mathfrak{h}_{\mathfrak{p}}^{\perp}, \mathfrak{h}_{\mathfrak{p}}^{\perp}\right] = 0$.

Cohomogeneity one foliations

- Irreducible case: Berndt, Tamaru.
- Reducible case: Berndt, Díaz-Ramos, Tamaru; Solonenko.

Horospherical type

$$\mathfrak{h} = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$$

$$\ell \subseteq \mathfrak{a}$$

Solvable type

$$\mathfrak{h} = \mathfrak{a} \oplus (\mathfrak{n} \ominus \ell)$$

$$\ell \subseteq \mathfrak{g}_\alpha, \alpha \in \Lambda$$

Foliations with section $\mathbb{R}^2$

- General classification: Berndt, Díaz-Ramos, Tamaru.
- $\Phi \subseteq \Lambda$ orthogonal subset.

$\Phi = \emptyset$

$(\mathfrak{a} \ominus V) \oplus \mathfrak{n}$,
$V \subseteq \mathfrak{a}$.

$\Phi = \{\alpha\}$

$(\mathfrak{a} \ominus V) \oplus (\mathfrak{n} \ominus \ell_\alpha)$,
$V \subseteq \ker \alpha$, $\ell_\alpha \subseteq \mathfrak{g}_\alpha$.

$\Phi = \{\alpha, \beta\}$

$\mathfrak{a} \oplus \big(\mathfrak{n} \ominus (\ell_\alpha \oplus \ell_\beta)\big)$,
$\ell_\lambda \subseteq \mathfrak{g}_\lambda$.

Foliations with section $\mathbb{R}H^2$

We outline the general strategy <u>for $\mathfrak{g}$ split</u>:

1. If $H \curvearrowright M$ induces a polar homogeneous foliation of codimension 2, then $\mathfrak{h}$ is solvable.

2. Structure of maximal solvable subalgebras (Mostow) ⇒ We can assume $\mathfrak{h} \subseteq \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{n}$ (split case).

3. Every subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$ induces a homogeneous foliation ⇒ apply the algebraic characterization.

Foliations with section $\mathbb{R}H^2$

Assume $\mathfrak{h} \subseteq \mathfrak{a} \oplus \mathfrak{n}$ induces a cohomogeneity two polar foliation.

1. The subspace $\tilde{\mathfrak{h}} = \mathfrak{h} + (\mathfrak{n} \ominus \mathfrak{n}^1)$ is a subalgebra.

2. Berndt, Tamaru $\Rightarrow \mathfrak{n} \ominus \mathfrak{n}^1 \subseteq \mathfrak{h} \Rightarrow \mathfrak{h}_{\mathfrak{p}}^{\perp} \subseteq \mathfrak{a} \oplus \mathfrak{p}^1$.

3. The normal space $(\mathfrak{a} \oplus \mathfrak{n}) \ominus \mathfrak{h}$ is spanned by H_α and $\xi \in \mathfrak{g}_\alpha$ for some $\alpha \in \Lambda$.

$$\langle H_\alpha, H \rangle = \alpha(H)$$

Foliations with section $\mathbb{R}H^2$

Theorem

Let $H \leq G$ be a closed subgroup inducing a codimention 2 polar homogeneous foliation on $M = G/K$. Then exactly one of the following assertions is true:

- $H \curvearrowright M$ is hyperpolar.
- There exists an $\alpha \in \Lambda$ such that $H \curvearrowright M$ is orbit equivalent to the action induced by the subgroup generated by

$$\mathfrak{h}_\alpha = (\mathfrak{a} \ominus \mathbb{R}H_\alpha) \oplus (\mathfrak{n} \ominus \mathfrak{g}_\alpha) = \ker\alpha \oplus (\mathfrak{n} \ominus \mathfrak{g}_\alpha).$$

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