Codimension two polar homogeneous foliations on symmetric spaces of noncompact type

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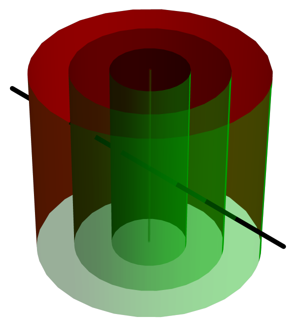
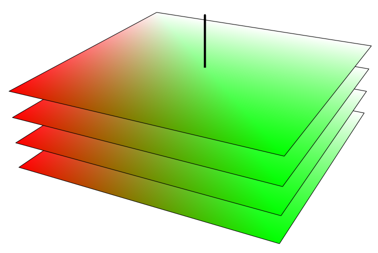
Introduction

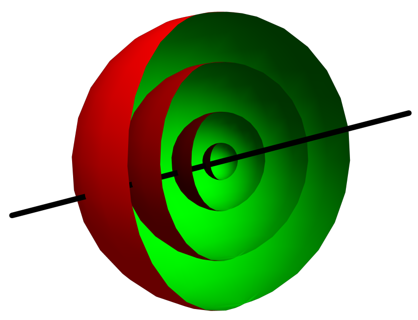
A proper isometric action of a Lie group G on a complete Riemannian manifold M is polar if there exists a complete submanifold $\Sigma \subseteq M$ meeting every orbit orthogonally.



$SO(2) \times \mathbb{R} \curvearrowright \mathbb{R}^3$



$\mathbb{R}^2 \curvearrowright \mathbb{R}^3$



$SO(3) \curvearrowright \mathbb{R}^3$

Main problem

Given M, classify all polar actions on M up to orbit equivalence.

The problem has been studied in manifolds with a large isometry group, such as symmetric spaces.
We focus on actions without singular orbits (polar homogeneous foliations) on symmetric spaces of noncompact type.

Previous results

- Codimension one foliations have been classified up to orbit equivalence in [2] and [4].
- A method for constructing all hyperpolar homogeneous foliations (that is, with $\Sigma = \mathbb{R}^n$) is described in [1], but the congruence problem remains open.

In codimension 2, the section is either $\mathbb{R}^2$ or homothetic to $\mathbb{R}H^2$. Our main objective is to study this last case. We will assume:

- $M = G/K$ is a symmetric space of noncompact type, $G = I^0(M)$, $K = G_o$ for a base point $o \in M$.
- $S \leq G$ is a connected closed subgroup acting polarly with codimension two orbits.

Iwasawa decomposition

The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is semisimple. Let $\mathfrak{p} = \mathfrak{k}^\perp$ (with respect to the Killing form), so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{p} = T_oM$. Choose a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$. Define

- For $\lambda \in \mathfrak{a}^*$, the root space

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \colon [H, X] = \lambda(H)X \ \forall H \in \mathfrak{a}\}.$$

- The set of roots

$$\Delta = \{\lambda \in \mathfrak{a}^* \colon \lambda \neq 0, \ \mathfrak{g}_\lambda \neq 0\}.$$

Given a subset of positive roots $\Delta^+ \subseteq \Delta$ and the corresponding simple roots Λ, let $\mathfrak{n} = \oplus_{\lambda \in \Delta^+} \mathfrak{g}_\lambda$.

Iwasawa decomposition

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ (as vector spaces).

The connected subgroup AN of G with Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$ acts simply transitively on M, so that $M \cong AN$. Every connected subgroup of AN induces a homogeneous foliation on M.

The classification

Choose $\alpha \in \Lambda$, a line $\ell \in \mathfrak{g}_\alpha$, and an abelian plane $\mathfrak{v} \subseteq \mathfrak{g}_\alpha$. Define the following subalgebras of $\mathfrak{a} \oplus \mathfrak{n}$:

$$\mathfrak{s}_\ell = \ker \alpha \oplus (\mathfrak{n} \ominus \ell)$$
$$\mathfrak{s}_\mathfrak{v} = \mathfrak{a} \oplus (\mathfrak{n} \ominus \mathfrak{v})$$

(Here, $\ominus$ denotes the orthogonal complement).
Take the connected subgroups $S_\mathfrak{v}$, S_ℓ with Lie algebras $\mathfrak{s}_\mathfrak{v}$, $\mathfrak{s}_\ell$.

Fact

$S_\mathfrak{v}$ and S_ℓ act polarly inducing a codimension 2 foliation with a non-flat section.

Main theorem

Any codimension 2 polar homogeneous foliation on M is hyperpolar or orbit equivalent to the action of an $S_\mathfrak{v}$ or S_ℓ.

Proof sketch

Suppose S induces a codimension 2 polar homogeneous foliation.

1. By [1], $\mathfrak{s}$ can be assumed to be solvable.
2. From a structure result on maximal solvable subalgebras of $\mathfrak{g}$ [3], we get $\mathfrak{s} \subseteq \mathfrak{z}_\mathfrak{k}(\mathfrak{a}) \oplus \mathfrak{a} \oplus \mathfrak{n}$.
3. By using the algebraic characterization of polar actions in [1], we obtain that $\mathrm{proj}_{\mathfrak{a} \oplus \mathfrak{n}}(\mathfrak{s})$ is equal to $\mathfrak{s}_\mathfrak{v}$ or $\mathfrak{s}_\ell$.
4. We prove that if $\mathrm{proj}_{\mathfrak{a} \oplus \mathfrak{n}}(\mathfrak{s})$ is $\mathfrak{s}_\mathfrak{v}$ (resp. $\mathfrak{s}_\ell$), then S has the same orbits as $S_\mathfrak{v}$ (resp. S_ℓ).

References

[1] J. Berndt, J. C. Díaz-Ramos, H. Tamaru, Hyperpolar homogeneous foliations on symmetric spaces of noncompact type, *J. Differential Geom.* **86** (2012), no. 2, 191-235.

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[3] G. D. Mostow, On maximal subgroups of real Lie groups, *Ann. of Math. (2)* **74** (1961), 503-517.

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